Semisimplicity and discriminator term in bounded BCK-algebras

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Dedicated to Roberto Cignoli

Abstract

By means of an adaption of a proof given by T.Kowalski in [15], we show that a relative subvariety of bounded BCK-algebras has all its members semisimple (i.e, it is a semisimple relative subvariety) if and only if it is a variety and verifies the identity

$$(x \to y) \to ((x^n \to \mathbf{0}) \to y) \to y) \approx \mathbf{1}, \text{ for some } n > 0.$$

Moreover, we show that discriminator varieties of bounded BCK-algebras are just the semisimple and involutive relative subvarieties. We also explain how these results can be extended to "strong expansions'" of the quasivariety of bounded BCK-algebras.

Key words : Bounded BCK-algebras, Semisimple relative subvariety, discriminator variety, strong expansion.

MSC (2000) : 03G25,03C05, 06F35, 08B99, 08C15.

Introduction

In [15] T. Kowalski shows that a variety R of *bounded, commutative and integral residuated lattices* (***Flew-algebras*** to short) has all its members semisimple (semisimple variety) if and only if it is a discriminator variety. To see this, he shows that any variety of semisimple Flew-algebras is n-contractive for some $n > 0$, i.e. $\mathsf{R} \models x^n \approx x^{n+1}$, and from this he deduces the following result:

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Theorem 0.1 [15, Theorem 3.10] *For a variety* R *of Flew-algebras the following conditions are equivalent:*

(i) R *verifies* $x \vee \neg x^n \approx \mathbf{1}$*, for some* $n \in \omega$

(ii) R *is a semisimple variety*

(iii) R *is a discriminator variety.*

Kowalski's proof only uses monoidal product, arrow, negation, and congruence permutability. Since in the above theorem the identity of item (i) can be replaced by the $\{\rightarrow, \mathbf{0}, \mathbf{1}\}$-identity

$$(x \rightarrow y) \rightarrow ((x^n \rightarrow \mathbf{0}) \rightarrow y) \rightarrow y) \approx \mathbf{1},$$

then our aim is to adapt this proof for bounded BCK-algebras avoiding the monoidal product and congruence permutability.

Bounded BCK-algebras were introduced by Iseki in [13] as BCK-algebras with an additional constant $\mathbf{0}$ interpreted as the lower bound. In fact, they are the algebraic counterpart of the BCK-logic plus a negation satisfying the Duns Scoto law. Since this can be expressed by means of a simple axiom, bounded BCK-logic is algebraizable (in the sense of Blok-Pigozzi [1]) and its equivalent algebraic semantics is the class of bounded BCK-algebras. Bounded BCK algebras are also the $\{\rightarrow, \mathbf{0}, \mathbf{1}\}$-subreducts of FLew-algebras.

The class bBCK of all bounded BCK-algebras is not a variety, but it is a quasivariety (see for example [22]). It is relatively congruence distributive and relatively $\mathbf{1}$-regular, that is, any bBCK-congruence is uniquely determined by its $\mathbf{1}$-equivalence class, which is an implicative filter. In fact, in any bounded BCK-algebra the family of its implicative filters is order isomorphic to the family of bBCK-congruences, both ordered by inclusion (see section 2 for details). This allows us to simplify the proof of [15, Lemma 2.1 and Theorem 2.7] which are previous to the proof of Theorem 0.1.

The paper has been organized in three sections. Section 1 is divided in two subsections; in Subsection 1.1, we give a definition of bounded BCK-algebra and we introduce some auxiliary $\{\rightarrow, \mathbf{0}, \mathbf{1}\}$-terms which will allow us to simplify some arithmetic calculations; and in Subsection 1.2, we give the results on implicative filters and maximal congruences needed in the paper. Section 2 contains our main result and some previous results. Finally, in Section 3 we see how these results can be extended to strong expansions of bBCK, that is quasivarieties in an expanded language, in which the added operations are bBCK-compatible and the class bBCK is precisely the class of all $\{\to, \mathbf{0}, \mathbf{1}\}$-subreducts; in the same way as bounded Pocrims, bounded BCK semilattices/lattices and Flew-algebras. For this, firstly, in subsection 3.1, we will show some general results on strong expansions of a quasivariety; and in subsection 3.2, we will apply these results to the quasivariety of bounded BCK-algebras.

We assume familiarity with basic concepts of universal algebra (see for instance [5]). There is a great deal of literature on BCK-algebras, but the references given by W. Blok and J. G. Raftery in [2, 3] are sufficiently representative. For arithmetical properties on bounded BCK-algebras we also use results given in [6] and [9].

1 Bounded BCK algebras. Preliminary results.

1.1 Some Arithmetical properties.

An algebra $\boldsymbol{\mathcal{A}} = \langle A; \to, \mathbf{0}, \mathbf{1}\rangle$ of type $(2,0,0)$ is called ***bounded BCK-algebra*** (***bBCK-algebra*** for short) provided that the following identities and quasi-identities hold in it:

(1) $(x \to y) \to ((y \to z) \to (x \to z) \approx \mathbf{1}$

(2) $\mathbf{1} \to x \approx x$

(3) $x \to \mathbf{1} \approx \mathbf{1}$

(4) $\mathbf{0} \to x \approx \mathbf{1}$

(5) $(x \to y \approx \mathbf{1})\,\&\,(y \to x \approx \mathbf{1}) \Rightarrow x \approx y$

The $\{\to, \mathbf{1}\}$-reducts of bounded BCK-algebras are BCK-algebras, which are defined by (1), (2), (3) and (5). The following properties are well known and can be found in the literature on BCK-algebras.

Lemma 1.1 *Every BCK-algebra* $\boldsymbol{\mathcal{C}} = \langle C; \to, \mathbf{1}\rangle$ *satisfies:*

(6) $x \to x \approx \mathbf{1}$

(7) $x \to (y \to x) \approx \mathbf{1}$

(8) $x \to (y \to z) \approx y \to (x \to z)$

(9) $y \to x \approx ((y \to x) \to x) \to x$

(10) $x \to ((x \to y) \to y) \approx \mathbf{1}$

Moreover, the relation

(11) $x \leq y$ *if and only if* $x \to y \approx \mathbf{1}$

is a partial order on C, *called the* ***natural order*** *of* $\boldsymbol{\mathcal{C}}$. □

If we consider the $\{\to, \mathbf{0}\}$-term $\neg x := x \to \mathbf{0}$, then (see [6, Lemma 1.6]):

Lemma 1.2 *The following properties hold true in any bounded BCK-algebra:*

a) $x \leq y \Rightarrow \neg y \leq \neg x$.

b) $\neg x \approx \neg\neg\neg x$.

c) $x \to \neg\neg x \approx \mathbf{1}$.

d) $x \to \neg y \approx y \to \neg x$.

e) $x \to \neg y \approx \neg\neg x \to \neg y$.

f) $\neg\neg(x \to \neg y) \approx x \to \neg y$.

g) $\neg x \to (x \to y) \approx \mathbf{1}$ $\square$

We represent by **BCK** the class of all BCK-algebras, and by **bBCK** the class of all bounded BCK-algebras; by definition, both are quasivarieties, but they are not varieties (see [21] and [22]).

If we consider the following $\{\to, \mathbf{0}\}$-terms,

$$\begin{array}{rcl} x^0 \to y & =: & y \\ x^{n+1} \to y & = & x \to ((x^n \to y)) \\ 0(x) & =: & \mathbf{0} \\ k(x) & =: & (x \to \mathbf{0})^k \to \mathbf{0} = (\neg x)^k \to \mathbf{0} \\ k(x^n) & =: & (x^n \to \mathbf{0})^k \to \mathbf{0} \\ x \oplus y & =: & \neg x \to \neg\neg y, \end{array}$$

where n, k are non-negative integers, then it is straightforward to check the arithmetical properties given in the following lemma.

Lemma 1.3 *In any bounded BCK-algebra the interpretation of the term operation* $+$ *is associative, commutative, has* $\mathbf{1}$ *as neutral element and* $\mathbf{0}$ *as absorbent element. Moreover, for any* k, n, r *positive integers, the following holds in any bounded BCK-algebra.*

(t1) $1(x) \approx 1(x^1) \approx \neg\neg x$

(t2) $x^n \to \mathbf{0} \approx (1(x^n)) \to \mathbf{0}$,
$x^{rn} \to \mathbf{0} \approx (1(x^r))^n \to \mathbf{0}$

(t3) $k(x^n) \approx k(1(x^n)) \approx 1(k(x^n))$

(t4) $x \le y \Rightarrow \begin{cases} y^n \to z \le x^n \to z \\ k(x^n) \le k(y^n) \\ x^n \to 1(y^n) \approx (1(x^n)) \to 1(y^n) \approx \mathbf{1} \end{cases}$

(t5) $k(x^{n+1}) \le k(x^n)$

(t6) $1(x^n) \le k(x^n) \le (k+1)(x^n) \approx k(x^n) \oplus 1(x^n)$

(t7) $(k+r)(x^n) \approx k(x^n) \oplus r(x^n)$. □

1.2 Implicative filters and congruences.

Congruence relations are always compatible with constants, hence for every $\boldsymbol{\mathcal{A}} \in \textsf{\textbf{bBCK}}$, $Con(\boldsymbol{\mathcal{A}}) = Con(\langle A; \to, \mathbf{1}\rangle)$. Since the quasivariety $\textsf{\textbf{BCK}}$ is relatively $\mathbf{1}$-regular, then $\textsf{\textbf{bBCK}}$ is also relatively $\mathbf{1}$-regular. In fact, for any bounded BCK-algebra $\boldsymbol{\mathcal{A}}$ we can consider $\mathcal{F}_i(\boldsymbol{\mathcal{A}})$ the family of all ***implicative-filters*** (***i-filters***) of $\boldsymbol{\mathcal{A}}$, i.e., the subsets F of A such that $1 \in F$, and $a, a \to b \in F$ implies $b \in F$. Then the correspondence $\theta \mapsto \mathbf{1}/\theta$ gives an order isomorphism from $Con_{\textsf{\textbf{bBCK}}}(\boldsymbol{\mathcal{A}}) = \{\theta \in Con(\boldsymbol{\mathcal{A}}) : \boldsymbol{\mathcal{A}}/\theta \in \textsf{\textbf{bBCK}}\}$ onto $\mathcal{F}_i(\boldsymbol{\mathcal{A}})$, both ordered by inclusion. Its inverse is given by $F \mapsto \theta_F = \{\langle a, b\rangle \in A \times A : a \to b, b \to a \in F\}$.

We shall recall some properties of i-filters in bounded BCK-algebras. In what follows, $\boldsymbol{\mathcal{A}}$ will represent a bounded BCK-algebra. For each $B \subseteq A$,

$$\langle B\rangle = \{a \in A : b_1 \to (\cdots \to (b_n \to a)) = \mathbf{1}, b_1, \ldots, b_n \in B,\ 1 \le n\}$$

is the *least implicative filter containing* B. In particular for any $a \in A$,

$$\langle\{a\}\rangle = \{b \in A : a^n \to b = \mathbf{1},\ \text{for some } n \ge 0\}$$

For short we write $\langle a\rangle$ in place of $\langle\{a\}\rangle$. The family $\mathcal{F}_i(\boldsymbol{\mathcal{A}})$ is closed under arbitrary intersections and under union of upward $\subseteq$-directed families, hence it forms an algebraic lattice that is distributive (see [16]), furthermore from Lemma 1.3 we deduce

Lemma 1.4 *For any $a \in A$, and any $n, k > 0$ the following properties hold:*

(f1) $\langle k(a^n)\rangle \subseteq \langle a\rangle$

(f2) $\langle (k+1)(a^n)\rangle \subseteq \langle k(a^n)\rangle$

(f3) $\langle k(a^n)\rangle \subseteq \langle k(a^{n+1})\rangle$. □

If for any $0 < k \in \omega$ and any $a \in A$ we consider:

$$F_k(a) = \bigcup_{n<\omega} \langle k(a^n) \rangle,$$

then $F_{k+1}(a) \subseteq F_k(a) \subseteq \langle a \rangle$. Therefore

$$F(a) = \bigcap_{1 \le k} F_k(a) \subseteq F_k(a) \subseteq \langle a \rangle.$$

We represent by $Max(\boldsymbol{\mathcal{A}})$ the set of maximal elements (***maximal i-filters***) of $\langle \mathcal{F}_i(\boldsymbol{\mathcal{A}}) \smallsetminus \{A\}, \subseteq \rangle$. The following result is well known.

Lemma 1.5 *Given $M \in \mathcal{F}_i(\boldsymbol{\mathcal{A}})$; then $M \in Max(\boldsymbol{\mathcal{A}})$ if and only if*

- *for each $a \in A$, $a \notin M$ iff there is $n > 0$ such that $a^n \to \mathbf{0} \in M$.* $\square$

Moreover, since

Lemma 1.6 *(see [8] for example) For any $\theta \in Con(\boldsymbol{\mathcal{A}})$ the following are equivalent*

(i) θ is maximal in $\langle Con_{\mathsf{bBCK}}(\boldsymbol{\mathcal{A}}) \smallsetminus \{\nabla_{\boldsymbol{\mathcal{A}}}\}, \subseteq \rangle$

(ii) θ is maximal in $\langle Con(\boldsymbol{\mathcal{A}}) \smallsetminus \{\nabla_{\boldsymbol{\mathcal{A}}}\}, \subseteq \rangle$

(iii) *$\theta = \theta_M$ for some $M \in Ci(\boldsymbol{\mathcal{A}})$.* $\square$

We say that $\boldsymbol{\mathcal{A}}$ is ***simple*** if it is non-trivial and $Con(\boldsymbol{\mathcal{A}}) = \{\Delta_A, \nabla_A\}$. Thus

Corollary 1.7 *$\boldsymbol{\mathcal{A}}$ is simple if and only if for any $a \in A \smallsetminus \{\mathbf{1}\}$ there is $n > 0$ such that $a^n \to \mathbf{0} = \mathbf{1}$.* $\square$

For each $\boldsymbol{\mathcal{A}} \in \mathsf{bBCK}$ we consider $Rad(\boldsymbol{\mathcal{A}}) = \bigcap Max(\boldsymbol{\mathcal{A}})$, then it is well known that

$$a \in Rad(\boldsymbol{\mathcal{A}}) \text{ iff for all } n \text{ there exists } k > 0 \text{ s.t. } k(x^n) = \mathbf{1}$$

Corollary 1.8 *For any $F \in \mathcal{F}_i(\boldsymbol{\mathcal{A}})$, $Rad(\boldsymbol{\mathcal{A}})/F \subseteq Rad(\boldsymbol{\mathcal{A}}/F)$.* $\square$

Since ***semisimple algebras*** are, by definition, non-trivial algebras isomorphic to a subdirect product of simple algebras, from Lemma 1.6 we also deduce

Corollary 1.9 *$\boldsymbol{\mathcal{A}}$ is semi-simple if and only if $Rad(\boldsymbol{\mathcal{A}}) = \{\mathbf{1}\}$.* $\square$

Remark 1.10 *It follows from the above results that the class* bBCK *is* ***hereditarily semisimple****, that is, the class of all semisimple bounded BCK-algebras is closed under subalgebras*

For further applications (see Lemma 2.6 below), we will consider some special cases of subquasivarieties of bBCK. First, we recall that by a *relative subvariety* of a quasivariety Q we understand any class V such that $\mathsf{V} = \mathsf{W} \cap \mathsf{Q}$ for some variety W (see [17] and [7]). If $\mathcal{H}$ is the operator homomorphic images, then a subquasivariety V of Q is a relative subvariety if and only if $\mathcal{H}(\mathsf{V}) \cap \mathsf{Q} \subseteq \mathsf{V}$. As a matter of fact, a relative subvariety can be defined by adding identities to an axiomatic basis of Q. Moreover, the condition "to be relative subvariety of" is transitive.

We also recall that the bounded BCK-algebras relatively subdirectly bBCK-irreducible are subdirectly irreducible in the absolute sense (see for instance [4] and [20]), hence in any relative subvariety V of bBCK relatively subdirectly V-irreducible algebras are subdirectly irreducible.

Given $k > 0$, a relative subvariety V of bBCK is called k-Rad, provided that for any $\boldsymbol{\mathcal{A}} \in \mathsf{V}$ we have:

$$Rad(\boldsymbol{\mathcal{A}}) = \{a \in A : \text{ for any } n > 0,\ k(x^n) = \mathbf{1}\}.$$

For instance, the relative subvariety SBCK (of all *Stonian BCK-algebras*) given by the equation

$$(\neg x \to y) \to ((\neg\neg x \to y) \to y) \approx \mathbf{1},$$

is 1-Rad; the relative subvariety $\neg\mathsf{BCK}$ given by the equation

$$((x \to \neg x) \to z) \to (((\neg x \to x) \to z) \to z) \approx \mathbf{1}$$

is 2-Rad. The relative subvariety bLBCK of all bounded BCK-algebras isomorphic to a subdirect product of bounded BCK-chains is given by the equation:

$$((x \to y) \to z) \to (((y \to x) \to z) \to z) \approx \mathbf{1};$$

it is a relative subvariety of $\neg\mathsf{BCK}$, and so it is 2-rad.

Remark 1.11 *From [9, Corollary 1.5] we deduce that if* $\boldsymbol{\mathcal{A}}$ *is a subdirectly irreducible bounded BCK-algebra, then for any* $a, b \in A$ *we have*

- *If* $\boldsymbol{\mathcal{A}} \in \mathsf{SBCK}$*, then* $\neg a = 1$ *or* $\neg\neg a = 1$

- *If* $\boldsymbol{\mathcal{A}} \in \textsf{\textbf{bLBCK}}$*, then* $a \leq b$ *or* $b \leq a$.
- *If* $\boldsymbol{\mathcal{A}} \in \neg\textsf{\textbf{BCK}}$*, then* $a \leq \neg a$ *or* $\neg a \leq a$.

Given $r, s > 0$, we consider **rsBCK** the relative subvariety of **bBCK** given by the equation

$$((x^{r+s} \to \mathbf{0}) \to y) \to ((((x^s \to \mathbf{0}) \to \neg(x^r \to \mathbf{0})) \to y) \to y) \approx \mathbf{1},$$

That is, for any $\boldsymbol{\mathcal{A}}$ finitely subdirectly irreducible of **rsBCK** and for any $a \in A$, one has $\neg(a^s \to \mathbf{0}) \leq a^r \to \mathbf{0}$ ($a^{r+s} \to \mathbf{0} = \mathbf{1}$) or $a^s \to \mathbf{0} \leq \neg(a^r \to \mathbf{0})$. In particular, $\neg$**BCK** $\subseteq$ **11BCK**.

Lemma 1.12 *The relative subvariety* **rsBCK** *is* $(r+s)$*-Rad*

Proof: Let $\boldsymbol{\mathcal{A}} \in$ **rsBCK**. Thus $Rad(\boldsymbol{\mathcal{A}}) \supseteq \{a \in A : \forall n \ (r+s)(a^n) = \mathbf{1}\}$ holds in any bounded BCK-algebra.
To show the other inclusion, take $\boldsymbol{\mathcal{A}} \hookrightarrow_{ps} \prod \boldsymbol{\mathcal{A}}_i$ a subdirect representation of $\boldsymbol{\mathcal{A}}$ in subdirectly irreducible algebras of **rsBCK**. If $a \in Rad(\boldsymbol{\mathcal{A}})$, then, by Corollary 1.8, for all $i \in I$, $a(i) \in Rad(\boldsymbol{\mathcal{A}}_i)$. Fix $n > 0$, then $s(a(i)^n), r(a(i)^n) \in Rad(\boldsymbol{\mathcal{A}})$, and $\neg r(a(i)^n) \notin Rad((\boldsymbol{\mathcal{A}})$, hence $s(a(i)^n) \not\leq \neg r(a(i)^n)$ that is

$$(a(i)^n \to \mathbf{0})^s \to \mathbf{0} \not\leq ((a(i)^n \to \mathbf{0})^r \to \mathbf{0}) \to \mathbf{0} = \neg(a(i)^n \to \mathbf{0})^r \to \mathbf{0}),$$

and so

$$(r+s)(a(i)^n) = (a(i)^n \to \mathbf{0})^{(r+s)} \to \mathbf{0} = \mathbf{1}$$

Therefore, $(r+s)(a^n) = \mathbf{1}$, for all $n > 0$. □

2 Semisimple relative subvarieties of bBCK

We say that a class of algebras is ***semisimple*** provided that all its members are semisimple or trivial. It is clear that in semisimple quasivarieties, subdirectly irreducible algebras coincide with simple algebras. In particular, **bBCK** is not semisimple, but for any X, its $|X|$-free algebra $\mathfrak{F}_{\textsf{\textbf{bBCK}}}(X)$[1] is semisimple (see [8]), hence its generated variety is also generated by simple bounded BCK-algebras.

From now one **V** will represent a relative subvariety of **bBCK**. If $\textsf{\textbf{V}}_{sim}$ represents the class of all simple algebras in **V**, and $\mathcal{V}$ the operator generated variety, then we have (c.f. [15, Lemma 3.6])

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[1] If there is no confusion, we use the same notation for variables and for free generators

Lemma 2.1 *If $\mathcal{V}(\mathbf{V}) = \mathcal{V}(\mathbf{V}_{sim})$ and for any $n > 0$,*

$$\mathbf{V} \not\models x^n \to \mathbf{0} \approx x^{n+1} \to \mathbf{0},$$

then for all $n > 0$ there is a simple algebra $\boldsymbol{\mathcal{A}}_n$ and $a_n \in A_n$ such that $a_n^n \to \mathbf{0} < a_n^{n+1} \to \mathbf{0} = \mathbf{1}$.

Proof: Assume the opposite, that is, there is $n > 0$ such that for any simple algebra $\boldsymbol{\mathcal{A}}$ in $\mathbf{V}$ and any $a \in A$,

$$a^{n+1} \to \mathbf{0} = \mathbf{1} \text{ implies } a^n \to \mathbf{0} = \mathbf{1}.$$

If $b \in A \smallsetminus \{\mathbf{1}\}$, then there is $k > 0$ such that $b^{k-\mathbf{1}} \to \mathbf{0} < b^k \to \mathbf{0} = \mathbf{1}$. We shall prove that $b^n \to \mathbf{0} = \mathbf{1}$. If $k \leq n$, then $\mathbf{1} = b^k \to \mathbf{0} \leq b^n \to \mathbf{0}$. If $k > n$, consider $r = \min\{s > 0 : sn \leq k - 1 < k \leq s(n+1)\}$, then

$$b^{rn} \to \mathbf{0} \leq b^{k-1} \to \mathbf{0} < b^k \to \mathbf{0} = \mathbf{1} \leq b^{r(n+1)} \to \mathbf{0},$$

Then by (t2)

$$(1(b^r))^n \to \mathbf{0} < \mathbf{1} \leq (1(b^r))^{(n+1)} \to \mathbf{0},$$

But, $(1(b^r))^{(n+1)} \to \mathbf{0} = \mathbf{1}$ forces $(1(b^r))^n \to \mathbf{0} = \mathbf{1}$ which is a contradiction. Since $1^n \to \mathbf{0} = \mathbf{0} = 1^{n+1} \to \mathbf{0}$, it follows that any simple algebra in $\mathbf{V}$ satisfies $x^n \to \mathbf{0} \approx x^{n+1} \to \mathbf{0}$, and so this equation holds in $\mathbf{V}$. This contradicts the assumption. $\square$

If for any bounded BCK-algebra $\boldsymbol{\mathcal{A}}$ we represent by $Ci(\boldsymbol{\mathcal{A}})$ the family of all completely $\cap$-irreducible i-filters, then, since subdirectly irreducible algebras are quotients by completely $\cap$-irreducible congruences, we have

Lemma 2.2 *$\mathbf{V}$ is semisimple if and only if for any $\boldsymbol{\mathcal{A}} \in \mathbf{V}$, $Ci(\boldsymbol{\mathcal{A}}) = Max(\boldsymbol{\mathcal{A}})$.* $\square$

Any proper i-filter is the intersection of all proper completely $\cap$-irreducible i-filters containing it. Therefore, by the above corollary, in algebras belonging to semisimple relative subvarieties any proper i-filter is the intersection of all maximal i-filters which contain it.

Lemma 2.3 *If $\mathbf{V}$ is semisimple, then for any $\boldsymbol{\mathcal{A}} \in \mathcal{V}$, and for any $a \in A$, $F(a) = \langle a \rangle$, that is, for any $k > 0$, $F_k(a) = \langle a \rangle$.*

Proof: Assume that there is $k > 0$ such that $F_k(a) \neq \langle a \rangle$, then, by Lemma 2.2, there exists $M \in Max(\boldsymbol{\mathcal{A}})$, such that, $F_k(a) \subseteq M$ and $a \notin M$, then for any $n > 0$ $k(a^n) = (a^n \to \mathbf{0})^k \to \mathbf{0} \in M$, and so for all $n > 0$, $a^n \to \mathbf{0} \notin M$, hence, by Lemma 1.5, $a \in M$. That contradicts the assumption. □

Theorem 2.4 (c.f. [15, Lemma 3.3]) *If* V *is a semisimple, then:*

(P) *there is a non-decreasing sequence* $\langle n(k) \rangle_{k>0}$ *of positive integers such that for any* $k > 0$, *there is* l_k *such that* V *satisfies the following equations:*

(Pk$_m$) $\mathsf{V} \models k(x^m)^{l_k} \to x \approx \mathbf{1}$, *for any* $n(k) \leq m$

Proof: Let $\mathfrak{F}$ be the $\{x\}$-free algebra on V. By Lemma 2.3, for any positive integer k there is $n > 0$, such that $\langle k(x^n) \rangle = \langle x \rangle$. By (f3) $n(k) = \min\{n : \langle k(x^n) \rangle = \langle x \rangle\}$ exists, thus

$$x \in \langle k(x^{n(k)}) \rangle \Rightarrow \exists l_k > 0 \text{ such that } k(x^{n(k)})^{l_k} \to x = \mathbf{1}$$

Taking into account that x is the free generator of $\mathfrak{F}$, we have that (Pk$_{n(k)}$) holds in V. Using the properties (t4) and (t5) we obtain that in the above equation we can replace $n(k)$ with any $m \geq n(k)$ and so (Pk$_m$) holds. To end, observe that (f2) guarantees that $n(k) \leq n(k+1)$. □

Remark 2.5 *Observe that under the conditions of the preceding Theorem, for any* $k > 0$, *we can* ***fix*** l_k *by taking* $l_k = \min\{l : k(x^{n(k)})^l \to x = \mathbf{1}\}$.

We do not know if the converse of the above theorem is true, but we shall prove that it holds in some relative subvarieties of bBCK.

Lemma 2.6 *Let* V *be a* k*-Rad relative subvariety of* bBCK, *with* $k > 0$, *then* V *is semisimple if and only if there are* $n, l > 0$, *such that*

(P'k$_m$) $\mathsf{V} \models k(x^m)^l \to x \approx \mathbf{1}$, □

Proof: If V is semisimple, then it follows from Theorem 2.4 that it satisfies (P'k$_m$). Assume that (P'k$_m$) holds, if $\boldsymbol{\mathcal{A}} \in \mathsf{V}$ and $a \in A \setminus \{\mathbf{1}\}$, then there exist l and m such that $(k(a^m))^l \to a = \mathbf{1}$, and so $k(a^m) \neq \mathbf{1}$. Thus $a \notin Rad(\boldsymbol{\mathcal{A}})$, and so $Rad(\boldsymbol{\mathcal{A}}) = \{\mathbf{1}\}$. □

Let V be a semisimple relative subvariety of bBCK, such that $\mathsf{V} \not\models x^{n+1} \to \mathbf{0} \approx x^n \to \mathbf{0}$, for any $n > 0$. Consider $\langle n(k) \rangle_{k>0}$ the sequence given in Theorem 2.4. Since $\mathcal{V}(\mathsf{V}) = \mathcal{V}(\mathsf{V}_{sim})$, by Lemma 2.1, for all $k > 0$ we have a simple algebra $\boldsymbol{\mathcal{S}}_k \in \mathsf{V}$ and $a_k \in S_k$ such that $a_k^{n(k)} \to \mathbf{0} < a_k^{n(k)+1} \to \mathbf{0} = \mathbf{1}$. Take $\boldsymbol{b} \in \prod_{k>0} \boldsymbol{\mathcal{S}}_k$ such that $\boldsymbol{b}(k) = a_k^{n(k)} \to \mathbf{0} \neq \mathbf{1}$, for any $k > 0$.

Theorem 2.7 *Under the above conditions, for any $k > 0$ and any $r \geq k$ we have $(l_k k+1)(\boldsymbol{b}(r)^k) = \mathbf{1}$, where l_k is the given in Theorem 2.4 (see Remark 2.5).*

Proof: Since $a(r) \leq \boldsymbol{b}(r) = a(r)^{n(r)} \rightarrow \mathbf{0}$, and $n(k) \leq n(r)$, we have

$$k(a(r)^{n(r)}) \leq k(\boldsymbol{b}(r)^{n(r)}) \leq k(\boldsymbol{b}(r)^{n(k)}),$$

then, since $k(a(r)^{n(r)}) = \boldsymbol{b}(r)^k \rightarrow \mathbf{0}$, by ($\mathrm{Pk}_{n(r)}$) and (t4), we have

$$\mathbf{1} = \left[k(\boldsymbol{b}(r)^{n(r)})\right]^{l_k} \rightarrow \boldsymbol{b}(r) \leq \left[k(a(r)^{n(r)})\right]^{l_k} \rightarrow \boldsymbol{b}(r) = (\boldsymbol{b}(r)^k \rightarrow \mathbf{0})^{l_k} \rightarrow \boldsymbol{b}(r),$$

after (t2), (t3) and (t4), this implies

$$\begin{aligned} \mathbf{1} &= (\boldsymbol{b}(r)^k \rightarrow \mathbf{0})^{l_k k} \rightarrow 1(\boldsymbol{b}(r)^k) = (\boldsymbol{b}(r)^k \rightarrow \mathbf{0})^{l_k k} \rightarrow ((\boldsymbol{b}(r)^k \rightarrow \mathbf{0}) \rightarrow \mathbf{0}) \\ &= (\boldsymbol{b}(r)^k \rightarrow \mathbf{0})^{(l_k k)+1} \rightarrow \mathbf{0} = ((l_k k) + 1)(\boldsymbol{b}(r)^k). \end{aligned}$$

Theorem 2.8 *If* V *is a semisimple relative subvariety of* bBCK*, then there is $n > 0$ such that*

($\mathbf{e}_n$) $\mathsf{V} \models x^{n+1} \rightarrow \mathbf{0} \approx x^n \rightarrow \mathbf{0}$.

Proof: Assume that $\mathsf{V} \not\models x^{n+1} \rightarrow \mathbf{0} \approx x^n \rightarrow \mathbf{0}$ for all $n > 0$. Then we can consider a family of simple algebras $\{\boldsymbol{\mathcal{S}}_k \colon k > 0\}$ of V an $\boldsymbol{b} \in \prod_{k>\omega} \boldsymbol{\mathcal{S}}_k$ such that for any $r > 0$, $\boldsymbol{b}(r) \neq \mathbf{1}$ and for any $k \geq 1$ there is s_k $(= (l_k k) + 1)$, such that $\{r : s_k(\boldsymbol{b}(r)^k) = \mathbf{1}\}$ is a cofinite set. Thus in any ultraproduct $\boldsymbol{\mathcal{B}} = \prod_{k>\omega} \boldsymbol{\mathcal{S}}_k/\mathcal{U}$ of $\{\boldsymbol{\mathcal{S}}_k \colon k > 0\}$ by a non-principal ultrafilter $\mathcal{U}$, $\boldsymbol{b}/\mathcal{U} \in Rad(\boldsymbol{\mathcal{B}}) \smallsetminus \{\mathbf{1}\}$. Hence $\boldsymbol{\mathcal{B}}$ is a non-semisimple algebra in V, and this contradicts the assumption. Therefore ($\mathbf{e}_n$) holds for some $n > 0$. □

Now we can give the *main result of the paper*:

Theorem 2.9 *Let* V *be a relative subvariety of* V*. Then* V *is semisimple if and only if there is $n > 0$ such that*

($\mathbf{EM}_n$) $\mathsf{V} \models (x \rightarrow y) \rightarrow ((x^n \rightarrow \mathbf{0}) \rightarrow y) \rightarrow y) \approx \mathbf{1}$.

Moreover, if V *is semisimple, then it is a variety.*

Proof: Suppose that V is semisimple. Then there is $n \geq 1$, such that $(\mathbf{e}_n)$ holds. Therefore, for any $\boldsymbol{\mathcal{A}} \in \mathsf{V}_{sim}$, and any $a \in A$, if $a \neq 1$, then there is $k > 1$ such that $a^k \to \mathbf{0} = 1$, and so by $(\mathbf{e}_n)$ $a^n \to \mathbf{0} = \mathbf{1}$. Thus for any $a \in A$, $\sup\{a, a^n \to \mathbf{0}\} = \mathbf{1}$, by [9, Corollary 1.5], we have that for any $b \in A$, $(a \to b) \to ((a^n \to \mathbf{0}) \to b) \to b) = \mathbf{1}$. Therefore $\mathsf{V}_{sim} \models (x \to y) \to ((x^n \to \mathbf{0}) \to y) \to y) \approx \mathbf{1}$, and so $(\mathbf{EM}_n)$ holds true. Conversely, if $(\mathbf{EM}_n)$ holds, then by Corollary 1.5 of [9] for any subdirectly irreducible $\boldsymbol{\mathcal{A}} \in \mathsf{V}$, condition $(\mathbf{EM}_n)$ implies that for any $a \in A$ $\sup\{a, a^n \to \mathbf{0}\} = \mathbf{1}$, and so $\boldsymbol{\mathcal{A}}$ is simple. Hence V is semisimple.
Finally, it is shown in the proof of Theorem 6.6 in [9], that $(\mathbf{EM}_n)$ implies

$(\mathbf{E}_n)$ $\mathsf{V} \models x^{n+1} \to y \approx x^n \to y$

Thus V is a variety (see [2, Proposition 13] for example). $\square$

We recall that a discriminator variety is a variety W generated by a class K of algebras having a common discriminator term, that is, a ternary term $d(x, y, z)$ in the language of W such that for any $\boldsymbol{\mathcal{A}} \in \mathsf{K}$ and any $a, b, c \in A$,

$$d^{\boldsymbol{\mathcal{A}}}(a, b, c) = \begin{cases} a, & \text{if } a \neq b \\ c, & \text{if } a = b. \end{cases}$$

It is well known that discriminator varieties are arithmetical and semisimple. In [9] it is shown that the variety generated by the simple algebra $\boldsymbol{\mathcal{E}} = \langle\{\bot, a', e, a, \top\}, \rightharpoonup, \mathbf{0}, \mathbf{1}\rangle$, given by the table

$\rightharpoonup$	$\mathbf{0}$	a'	e	a	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a'	a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
e	a'	a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	a'	a	a	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a'	e	a	$\mathbf{1}$

,

is not congruence permutable. Thus $\mathcal{V}(\boldsymbol{\mathcal{E}})$ is a relative subvariety of **bBCK** satisfying (EM$_3$) which is not a discriminator variety.

In general we can establish a result (reported to me by Joan Gispert) that characterizes discriminator varieties of bounded BCK-algebras. This result also answers the last question proposed in [9]. For this we need a previous result similar to the one given in [18] and [2] for BCK-algebras and easily provable by induction on the complexity of terms.

Lemma 2.10 *Let $t(x_1,\ldots,x_n)$ be a $\{\to,\mathbf{0},\mathbf{1}\}$-term, then there are $m>0$ and $\alpha_1(x_1,\ldots,x_n),\ldots,\alpha_m(x_1,\ldots,x_n)$ $\{\to,\mathbf{0}\}$ terms such that one of the following properties hold*

(td1) $\mathsf{bBCK} \models t(x_1,\ldots,x_n) \approx \alpha_1 \to (\cdots \to (\alpha_m \to x_i))$ *for some* $1 \leq i \leq n$

(td2) $\mathsf{bBCK} \models t(x_1,\ldots,x_n) \approx \alpha_1 \to (\cdots \to (\alpha_m \to \mathbf{0}))$

Remark 2.11 *Observe that if $t(x_1,\ldots,x_n)$ satisfies* **(td1)***, then*

$$\mathsf{bBCK} \models t(x_1,\ldots,x_{i-1},\mathbf{1},x_{i+1},\ldots,x_n) \approx \mathbf{1}.$$

Moreover, by item f) of Lemma 1.2, **(td2)** *holds, if and only if,*

$$\mathsf{bBCK} \models \neg\neg t(x_1,\ldots,x_n) \approx t(x_1,\ldots,x_n).$$

Theorem 2.12 V *is a discriminator variety, if and only if* V *is semisimple and* $\mathsf{V} \models \neg\neg x \approx x$.

Proof: Assume that V is a discriminator variety, that is, there is $\mathsf{K} \subseteq \mathsf{V}$, such that $\mathsf{V} = \mathcal{V}(\mathsf{K})$ and there is a ternary $\{\to,\mathbf{0}\}$-term $d(x,y,z)$, such that for any $\boldsymbol{\mathcal{A}} \in \mathsf{K}$, and $a,b,c \in A$, $d^{\boldsymbol{\mathcal{A}}}(a,a,c)=c$ and if $a \neq b$, $d(a,b,c)=a$. Since,

$$d^{\boldsymbol{\mathcal{A}}}(\mathbf{1},\mathbf{1},\mathbf{0}) = \mathbf{0} = d^{\boldsymbol{\mathcal{A}}}(\mathbf{0},\mathbf{1},\mathbf{1})$$

By the observations in Remark 2.11, we have that $d(x,y,z)$ satisfies **(td2)**, then

$$\mathsf{K} \models x \approx d(x,x,x) \approx \neg\neg d(x,x,x) \approx \neg\neg x$$

and so $\mathsf{V} \models x \approx \neg\neg x$.
The converse in given in [9, Corollary 6.7]. □

Bounded BCK-algebra satisfying the identity $\neg\neg x \approx x$ are called ***involutive*** BCK-algebras. They are term-wise equivalent to Grišin's L_0-algebras, also called involutive pocrims. The relative subvariety IBCK of all involutive BCK-algebras is not a variety (see for instance[10]).

3 Extending the results to strong expansions of bBCK

To extend Theorems 2.8 and 2.12 to some compatible expansions, in the sense of [19], we need to show some general results.

3.1 Strong expansions of a quasivariety

For any quasivariety Q, we represent its algebraic language by $L(\mathsf{Q})$. Following the preceding notation, for each $L(\mathsf{Q})$-algebra $\boldsymbol{\mathcal{A}}$ we will consider $Con_{\mathsf{Q}}(\boldsymbol{\mathcal{A}}) = \{\theta \in Con(\boldsymbol{\mathcal{A}}) : \boldsymbol{\mathcal{A}}/\theta \in \mathsf{Q}\}$ the family of all Q-congruences of $\boldsymbol{\mathcal{A}}$. If $L \subseteq L(\mathsf{Q})$, i.e., L is a sublanguage of $L(\mathsf{Q})$ or $L(\mathsf{Q})$ is an expansion of L, then $\boldsymbol{\mathcal{A}} \upharpoonright L$ represents the L-reduct of $\boldsymbol{\mathcal{A}}$. And we write $\mathsf{Q}\upharpoonright L = \{\boldsymbol{\mathcal{A}}\upharpoonright L : \boldsymbol{\mathcal{A}} \in \mathsf{Q}\}$. Then for any quasivariety Q the class $\mathcal{S}(\mathsf{Q}\upharpoonright L)$ of all L-subreducts of Q is a quasivariety.

By a ***compatible expansion*** of a quasivariety Q we understand a quasivariety R, such that $L(\mathsf{Q}) \subseteq L(\mathsf{R})$, i.e., the language of R is an expansion of the language of R, and it satisfies:

(CE) the operation terms in $L(\mathsf{R}) \smallsetminus L(\mathsf{Q})$ are Q-compatible, that is for any $\boldsymbol{\mathcal{B}} \in \mathsf{R}$, $Con_{\mathsf{R}}(\boldsymbol{\mathcal{B}}) = Con_{\mathsf{Q}}(\boldsymbol{\mathcal{B}}\upharpoonright L(\mathsf{Q}))$.

Observe that any relative subvariety can be regarded as a compatible expansion.

By a ***strong expansion*** of Q we understand a *compatible expansion* R, such that

(SC) Q is the class of all subreducts of R. That is $\mathsf{Q} = \mathcal{S}(\mathsf{R}\upharpoonright L(\mathsf{R}))$.

And we say that a quasivariety Q has the ***Relative congruence extension property*** when:

(RCEP) If $\boldsymbol{\mathcal{A}} \in \mathsf{Q}$ and any $\boldsymbol{\mathcal{C}} \subseteq \boldsymbol{\mathcal{A}}$, then for each $\theta \in Con_{\mathsf{Q}}(\boldsymbol{\mathcal{C}})$ there is $\theta' \in Con_{\mathsf{Q}}(\boldsymbol{\mathcal{B}})$ such that $\theta = \theta' \cap B^2$.

Lemma 3.1 *Let R be a strong expansion of the quasivariety Q. If Q has the RCEP, then for every V relative subvariety of R, we have*

1) *$\mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$ is a relative subvariety of Q.*

2) *if $\mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$ is variety , then V is also a variety.*

Proof: 1) Since $\mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$ is a subquasivariety of Q, then it is enough to see that $\mathcal{H}(\mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))) \cap \mathsf{Q} \subseteq \mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$. Let $\boldsymbol{\mathcal{A}} \in \mathcal{H}(\mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))) \cap \mathsf{Q}$. Then there is $\boldsymbol{\mathcal{B}} \in \mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$, and $\theta \in Con_{\mathsf{Q}}(\boldsymbol{\mathcal{B}})$ such that $\boldsymbol{\mathcal{B}}/\theta \cong \boldsymbol{\mathcal{A}}$. Let $\widehat{\boldsymbol{\mathcal{B}}} \in \mathsf{V}$ such that $\boldsymbol{\mathcal{B}} \subseteq \widehat{\boldsymbol{\mathcal{B}}}\upharpoonright L(\mathsf{Q})$. By RCEP there exists $\widehat{\theta} \in Con_{\mathsf{Q}}(\widehat{\boldsymbol{\mathcal{B}}}\upharpoonright L(\mathsf{Q}))$ such that $\theta = \widehat{\theta} \cap B^2$. Since by **(CE)** $\widehat{\theta} \in Con_{\mathsf{R}}(\widehat{\boldsymbol{\mathcal{B}}})$, then the correspondence $h\colon a/\theta \mapsto a/\widehat{\theta}$ gives an embedding from $\boldsymbol{\mathcal{B}}/\theta$ into $\widehat{\boldsymbol{\mathcal{B}}}/\widehat{\theta}\upharpoonright L(\mathsf{Q})$. Since $\widehat{\boldsymbol{\mathcal{B}}}/\widehat{\theta} \in \mathcal{H}(\mathsf{V}) \cap \mathsf{R} \subseteq \mathsf{V}$, we have $\boldsymbol{\mathcal{B}}/\theta \in \mathcal{S}(\mathsf{V}\upharpoonright L(\mathsf{Q}))$.

2) Assume that $\mathcal{S}(\mathsf{V}\restriction L(\mathsf{Q}))$ is a variety. If $\boldsymbol{\mathcal{A}} \in \mathsf{V}$, and $\theta \in Con(\boldsymbol{\mathcal{A}})$, since $Con(\boldsymbol{\mathcal{A}}) = Con_{\mathsf{Q}}(\boldsymbol{\mathcal{A}}\restriction L(\mathsf{Q})) = Con_{\mathsf{R}}(\boldsymbol{\mathcal{A}})$ we have that $\boldsymbol{\mathcal{A}}/\theta \in \mathcal{H}(\mathsf{V})\cap\mathsf{R} \subseteq \mathsf{V}$. Therefore V is closed under homomorphic images, and so it is a variety. □

Thus it is easy to show that:

Corollary 3.2 *Let* Q *be a hereditarily semisimple quasivariety i.e., the class of all semisimple algebras is closed under subalgebras, which has the RCEP. If* V *is a semisimple relative subvariety of a strong expansion* R *of* Q*, then* $\mathcal{S}(\mathsf{V}\restriction L(\mathsf{Q}))$ *is a semisimple relative subvariety of* Q.

3.2 Strong expansions of **bBCK**

Since the quasivariety **bBCK** is hereditarily semisimple and it has the RCEP, from Lemma 3.1, Corollary 3.2 and Theorem 2.9 we deduce

Theorem 3.3 *Let* R *be a strong expansion of* **bBCK**. *For any relative subvariety* V *of* R*, the following are equivalent*

- (i) V *is semisimple.*
- (ii) V *satisfies* $(\mathbf{EM}_n)$ *for some* $n > 0$.
- (iii) V *is a semisimple variety.*

From the results given in the literature on BCK-algebras we can give several strong expansions of the quasivariety **bBCK** (see for instance [11] [12], and [6]). For example, the class **bPo** of ***bounded Pocrims***. A bounded pocrim is an algebra $\boldsymbol{\mathcal{P}} = \langle P; \to, *, \mathbf{0}, \mathbf{1}\rangle$ of type $(2, 2, , 0, 0)$, such that

- its $\{\to, \mathbf{0}, \mathbf{1}\}$-reduct $\boldsymbol{\mathcal{P}}^- = \langle P; \to, \mathbf{0}, \mathbf{1}\rangle$ is a bounded BCK-algebra,
- it satisfies the equation:

  (S) $(x * y) \to z \approx x \to (y \to z)$

It is well known (see for instance [11, 12, 6]) that the class **bPo** is a quasivariety which is not a variety, and it is a strong expansion of **bBCK**.

If $\mathsf{Q} \in \{\mathbf{bBCK}, \mathbf{bPo}\}$ and $\gamma \subseteq \{\wedge, \vee\}$, then Q^γ will represent the quasivariety of all $(L(\mathsf{Q}) \cup \gamma)$-algebras, such that

- for $\gamma = \{\wedge\}$ they satisfy bounded BCK axioms, semilattice equations and the natural partial order is the meet-semilattice partial order,

- for $\gamma = \{\vee\}$, they satisfy bounded BCK axioms, semilattice equations and the natural partial order is the join-semilattice partial order,
- for $\gamma = \{\wedge, \vee\}$, they satisfy bounded BCK equations, lattice equations and the natural partial order is the lattice partial order.

It follows from the results given in [11] and [12], that if $\gamma \neq \emptyset$, then Q^{γ} is strong expansion of Q and it is a variety. Moreover, $\mathsf{bPo}^{\{\wedge\vee\}}$ is precisely the class of all bounded commutative, integral, residuated lattices, i.e., the class **Flew**. Moreover, for all γ, Q^{γ} is a strong expansion of **bBCK**, but, as shown in [11], $\mathsf{bPo}^{\{\wedge\}}$ and **Flew** are not strong expansions of the class $\mathsf{bBCK}^{\{\wedge\}}$ because **(SC)** fails.

It is easy to see that in the classes **bPo**, $\mathsf{bBCK}^{\{\wedge\}}$, $\mathsf{bBCK}^{\wedge,\vee}$ and, of cors, the class **Flew**, semisimple subvarieties are discriminator varieties. This is not true in $\mathsf{bBCK}^{\vee}$. Take for example the algebra $\langle \boldsymbol{\mathcal{E}}, \vee\rangle$, obtained from $\boldsymbol{\mathcal{E}}$ given in page 12, adding the join operation given by the natural order. Then since $\langle \boldsymbol{\mathcal{E}}, \vee\rangle$ satisfies $(\mathbf{EM}_3)$, and it generates a semisimple subvariety of $\mathsf{bBCK}^{\vee}$ which is not congruence permutable (see the algebra $\boldsymbol{\mathcal{K}} \in \mathcal{V}(\boldsymbol{\mathcal{E}})$ given in [9]).

We can give a general result

Theorem 3.4 *Let* R *be a strong expansion of* **bBCK**, *then a semisimple relative subvariety* V *of* R *is a discriminator variety, iff there is a binary* $L(\mathsf{R})$*-term* $t(x, y)$ *such that:*

$(d1)$ $\mathsf{V} \models t(\mathbf{1}, x) \approx t(x, \mathbf{1}) \approx x$

$(d2)$ *for all* $\boldsymbol{\mathcal{A}} \in \mathsf{V}$ *and any* $a, b \in A$, $t^{\boldsymbol{\mathcal{A}}}(a, b) = \mathbf{1}$ *iff* $a = b = \mathbf{1}$.

Proof: The direct sense follows from the fact that if $d(x, y, z)$ is a discriminator term of an algebra $\boldsymbol{\mathcal{A}}$ the term $t(x, y) = d(x, \mathbf{1}, d(y, x, y))$ satisfies $(d1)$ and $(d2)$ for the variety generated by $\boldsymbol{\mathcal{A}}$.
Conversely, assume that there is $t(x, y)$ satisfying the $(d1)$ and $(d2)$, and let $n > 0$ such that V satisfies $(\mathbf{EM}_n)$. Then a simple calculation shows that:

$$t(t(x \to y, y \to x)^n \to z, (t(x \to y, y \to x)^n \to \mathbf{0}) \to x)$$

is a discriminator term on any simple algebra in V. □

Observe that in **bPo** the operation $*$ satisfies $(d1)$ and $(d2)$, as well as $\wedge$ in $\mathsf{bBCK}^{\wedge}$, $\mathsf{bBCK}^{\wedge,\vee}$. Finally, note that in **Flew** $*$ and $\wedge$ satisfy $(d1)$ and $(d2)$.

Observe also that from Remark 2.11 we can deduce that a relative subvariety of **bBCK** admits a term $t(x, y)$ satisfying $(d1)$ if and only if it is involutive, i.e., it satisfies the identity $x \approx \neg\neg x$.

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