Representations of structural closure operators

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Abstract

We continue the work of Blok and Jónsson by developing the theory of structural closure operators and introducing the notion of a representation between them. Similarities and equivalences of Blok-Jónsson are bijective representations and bijective structural representations, respectively. We obtain a characterization for representations induced by a translation. In order to obtain a similar characterization in the structural case we introduce the notions of a graduation and a graded variable of an M-set. We show that several deductive systems, Gentzen systems among them, are graded M-sets having graded variables, and describe the graded variables in each case. In the last section we show that, for a sentential logic, having an algebraic semantics is equivalent to being representable in an equational consequence. We prove that if a closure operator is representable by a translation, then every extension of it is also representable by the same translation. This generalizes a result of Blok and Rebagliato.

1 Introduction

In the study of logic, a number of different kinds of deductive systems have been designed for different purposes and some of them have been studied with algebraic techniques. Among these, we find sentential logics, equational consequences, k-dimensional deductive systems, Gentzen systems, m-sided sequent systems, Gentzen systems of hypersequents, ... All of them are based on the notion of *consequence*. This notion has two components: one is purely set-theoretic and can be formalized in the abstract notion of closure relations (and closure operators and systems). The other is its formal character, which is usually identified with the notion of substitution-invariance or *structurality*: the structurality property states that "consequence is invariant under uniform substitutions", that is, consequence depends only on the form of the sentences and not on their content.

The recent developments in Abstract Algebraic Logic have taken this formalization of consequence as the framework where in particular the notion of algebraizable logic has been identified and characterized (see [BP89]). Within this formalization, structurality requires to have an algebra of formulas and a set of "sentences" built up from formulas in some way, as in the mentioned deductive systems. Substitutions are then the endomorphisms of the formula algebra. The set of substitutions together with composition forms a monoid that acts by evaluation on the set of formulas. The sets of sequents, hypersequents, etc. inherit an action of the set of substitutions as well, and the structurality property for the deductive systems based on these sets is formulated in terms of this action. (See examples in Section 3.)

Blok and Jónsson (see [BJ06]) opened a new path to formalize structurality for *arbitrary* sets. The main idea in their formalization of the structurality property of a closure relation (operator, system) on a set is to generalize the monoid of substitutions to an arbitrary monoid that acts on this set. A set acted on by a monoid M is said to be an *M-set*. Closure operators on an M-set that are compatible with the action are called *structural* closure operators on the M-set. This generalization allows encompassing all the mentioned systems in a common study, simplifying and unifying proofs of similar theorems. In the present article we continue developing this theory.

In Section 4, we present translations between sets as residuated maps between their powersets. It is well known that for a closure operator C, the set of C-closed sets is a complete lattice. We explain how a translation τ from B to A induces a monotone map $(\cdot)^\tau$ between the lattice of closure operators on A and the lattice of closure operators on B. We introduce the notion of *τ-filter* of a closure operator on B and show that τ-filters form a closure system on A. We prove that this defines a map $(\cdot)_\tau$ from the lattice of closure operators on B to the lattice of closure operators on A, which is residuated and whose residuum is $(\cdot)^\tau$.

In Section 5, we introduce the notion of a (structural) representation of one (structural) closure operator in another. A representation of D in C is an embedding of the lattice of D-closed sets in the lattice of the C-closed sets. Similarities and equivalences of Blok-Jónsson appear then as bijective representations and bijective structural representations, respectively. We obtain a characterization for representations induced by a translation as those representations that preserve all joins, which we call *join* representations.

We show (by a counterexample) that a similar characterization of structural representations is not possible in general. We have to restrict ourselves to a certain class of M-sets in order to obtain the characterization. To this end, in Section 6, we introduce the notions of a graduation and a graded variable of an M-set. We show that sentential logics, equational logics, Gentzen systems, m-sided sequent systems and Gentzen systems of hypersequents are based on graded M-sets, and describe graded variables in each case. In Section 7, we prove that structural join representations of structural closure operators on graded M-sets having graded variables are those representations induced by a structural translation, and obtain as a corollary that equivalences between structural closure operators on graded M-sets having graded variables are always induced by structural translations.

In the last section we show that, for a sentential logic, having an algebraic semantics is equivalent to being representable in the equational consequence of a class of algebras. Thus, we can extend the notion of algebraic semantics to Gentzen systems, hypersequent systems, etc. Blok and Rebagliato [BR03] proved that if a sentential logic has an algebraic semantics, then each of its extensions also has an algebraic semantics with the same defining equations. We prove a generalization of this result in the following terms: if a structural closure operator is representable by a structural translation, then every structural extension of it is also representable by the same translation. As a consequence, if a Gentzen system, a hypersequent system, etc. has an algebraic semantics, then so does each of its extensions, and with the same defining equations.

For general references of Abstract Algebraic Logic, the reader will find the survey [FJP03] and the monograph [Cze01] useful, and the references therein. In [Bly05] the reader will find a detailed exposition and study of residuated maps.

2 Preliminaries

A *closure operator* on a set A is an expansive monotone idempotent map[1] $C : \mathcal{P}A \to \mathcal{P}A$, i.e., for every $X, Y \subseteq A$, C satisfies that $X \subseteq CX$, if $X \subseteq Y$ then $CX \subseteq CY$, and $CCX = CX$. A *closure system* on a set A is a subset of $\mathcal{P}A$ containing A and closed under arbitrary intersections. The closure system associated with a closure operator C is $\mathrm{Cl}(C) = \{T \subseteq A : CT = T\}$, and its elements are called the *C-closed sets* or *theories* of C. The closure operator associated with a closure system $\mathcal{C}$ is the map $C : \mathcal{P}A \to \mathcal{P}A$ such that for every $X \subseteq A$, $CX = \bigcap\{T \in \mathcal{C} : X \subseteq T\}$. These two correspondences are inverse of one another, and hence bijective.

Since closure systems are closed under arbitrary intersections, they have a structure of complete lattices, where the supremum of a set of C-closed sets $\{T_\lambda : \lambda \in \Lambda\}$ is $\bigvee_{\lambda\in\Lambda} T_\lambda = C \bigcup_{\lambda\in\Lambda} T_\lambda$. We will use the notation $\mathbf{Cl}(C) = \langle \mathrm{Cl}(C), \subseteq \rangle$.

We want to emphasize the following property that will be used often: for every family, $\{X_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{P}A$,

$$C \bigcup_{\lambda\in\Lambda} X_\lambda = C \bigcup_{\lambda\in\Lambda} CX_\lambda = \bigvee_{\lambda\in\Lambda} CX_\lambda \tag{1}$$

A *closure relation* on a set A is a binary relation $\vdash$ between subsets of A and elements of A such that, for every $a \in A$ and every $X, Y \subseteq A$,

(i) if $a \in X$, then $X \vdash a$,

(ii) if $Y \vdash x$ for every $x \in X$, and $X \vdash a$, then $Y \vdash a$.

If $X, Y \subseteq A$, then we write $X \vdash Y$ for $\forall a \in Y$, $X \vdash a$, and $X \dashv\vdash Y$ for $X \vdash Y$ and $Y \vdash X$.

The closure relation associated with a closure operator C is the relation $\vdash_C$ determined by

$$X \vdash_C a \quad \Leftrightarrow \quad a \in CX.$$

The closure operator determined by a closure relation $\vdash$ is the map $C^{\vdash} : \mathcal{P}A \to \mathcal{P}A$ defined as follows: for every $X \subseteq A$, $C^{\vdash}X = \{a \in A : X \vdash a\}$. These two correspondences are inverse of one another, and hence bijective.

[1] Unless any confusion arises, we will denote the image of an element a by a function f by fa. Accordingly, we will denote composition of maps by juxtaposition.

The set of closure operators on a set can be ordered in the following way: if C and C' are closure operators on a set A, then $C \leqslant C'$ when for every $X \subseteq A$, $CX \subseteq C'X$. In that case we say that C' is an *extension* of C. This order renders the set $\mathrm{Clop}(A)$ of closure operators on A a complete lattice, which we denote by $\mathbf{Clop}(A)$. Closure systems and relations are ordered by inclusion. We have the equivalences:

$$C \leqslant C' \quad \Leftrightarrow \quad \mathrm{Cl}(C') \subseteq \mathrm{Cl}(C) \quad \Leftrightarrow \quad \vdash_C \subseteq \vdash_{C'}.$$

Given two orders $\langle A, \leqslant \rangle$ and $\langle B, \leqslant' \rangle$, a map $f : A \to B$ is called *residuated* if there exists a map $f^* : B \to A$, called the *residuum* of f, such that for every $a \in A$ and every $b \in B$, the following biconditional is satisfied:

$$a \leqslant f^* b \quad \Leftrightarrow \quad fa \leqslant' b.$$

We also say that $\langle f, f^* \rangle$ is a *residuated pair*. The following is a well-known result for residuated maps:

Lemma 1. *If $f : A \to B$ and $g : B \to C$ are residuated maps (with respect to certain orders in A, B and C), then*

(i) f preserves all joins and f^* preserves all meets.

(ii) $id_A \leqslant f^* f$ and $f f^* \leqslant id_B$.

(iii) gf is residuated and its residuum is $(gf)^* = f^* g^*$.

Finally, recall that a *monoid* is a set M endowed with an associative binary operation, which we will denote by juxtaposition, and a constant 1, the *unit* of M, which is neutral with respect to the operation of M.

3 Actions and structural closure operators

Consequence relations are introduced in the literature for several kinds of "sentences", for instance, propositional formulas, equations, sequents, m-sided sequents, hypersequents, etc. They are closure relations that satisfy a property of "formality", often called *structurality*: consequence is invariant under uniform substitutions. This can be expressed as follows: for every set of "sentences" $X \cup \{x\}$ and every substitution σ,

$$X \vdash x \quad \Rightarrow \quad \sigma[X] \vdash \sigma(x).$$

Following [BJ06] we give an abstract notion of structurality for closure operators on arbitrary sets. It is based on the fact that structurality for the consequence relations of logics in the usual sense involves the set of substitutions, i.e., the set $\mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ of endomorphisms of the algebra of formulas $\mathbf{Fm}_{\mathcal{L}}$, for some language $\mathcal{L}$, which is a monoid that acts on the set of "sentences".

Definition 2. Let M be a monoid. An *M-set* is a set A together with an *action* of M *on* A, that is, a map $\cdot : M \times A \to A$ such that, for every $\sigma, \sigma' \in M$ and every $x \in A$,

(i) $(\sigma\sigma') \cdot x = \sigma \cdot (\sigma' \cdot x)$,

(ii) $1 \cdot x = x$.

A *constant* of an M-set is an element $a \in A$ such that for every $\sigma \in M$, $\sigma \cdot a = a$.

If M is a monoid and $\langle A, \cdot \rangle$ is an M-set, we also denote the operation $\cdot$ with juxtaposition. Furthermore, we define for each $\sigma \in M$ the maps $\sigma_A, \sigma_A^{-1} : \mathcal{P}A \to \mathcal{P}A$ determined by:

$$\sigma_A X = \{\sigma x : x \in X\} \qquad \sigma_A^{-1} X = \{x \in A : \sigma x \in X\}.$$

Note that σ_A is a residuated map and that σ_A^{-1} is not the set-theoretic inverse of σ_A, but its residuum $(\sigma_A)^*$, since for every $X, Y \subseteq A$,

$$Y \subseteq \sigma_A^{-1} X \quad \Leftrightarrow \quad \sigma_A Y \subseteq X.$$

Unless any confusion comes up, we omit the subscripts.

Definition 3. Let M be a monoid and $\langle A, \cdot \rangle$ an M-set. A closure operator C on A (and its associated closure system and closure relation) is *M-structural on* $\langle A, \cdot \rangle$, and the action is said to be *C-compatible*, when for every $\sigma \in M$,

$$\sigma C \leqslant C\sigma,$$

that is to say, for every $X \subseteq A$ and every $x \in CX$, $\sigma x \in C\sigma X$.

Note that, if $\langle A, \cdot \rangle$ is an M-set, $\vdash$ is a closure relation on A and $C^{\vdash}$ is its associated closure operator, then the structurality condition for $C^{\vdash}$ can be expressed in the following way: for every $X \subseteq A$, $a \in A$, and $\sigma \in M$,

$$X \vdash a \quad \Rightarrow \quad \sigma X \vdash \sigma a.$$

Some of the following properties appear in [BJ06] as consequences of structurality, but in fact they are equivalent to it:

Lemma 4. *If M is a monoid, $\langle A, \cdot \rangle$ is an M-set, C is a closure operator on A and $\sigma \in M$, then the following statements are equivalent:*

(i) $\sigma C \leqslant C\sigma$;

(ii) $C\sigma C = C\sigma$;

(iii) $\sigma C \sigma^{-1} \leqslant C$;

(iv) $C\sigma^{-1} \leqslant \sigma^{-1} C$;

(v) $C\sigma^{-1} C = \sigma^{-1} C$.

Proof.

$(i) \Rightarrow (ii)$ $C\sigma C \leqslant CC\sigma = C\sigma \leqslant C\sigma C$.

$(ii) \Rightarrow (iii)$ $\sigma C\sigma^{-1} \leqslant C\sigma C\sigma^{-1} = C\sigma\sigma^{-1} \leqslant C$.

$(iii) \Rightarrow (iv)$ $C\sigma^{-1} \leqslant \sigma^{-1}\sigma C\sigma^{-1} \leqslant \sigma^{-1}C$.

$(iv) \Rightarrow (v)$ $C\sigma^{-1}C \leqslant \sigma^{-1}CC = \sigma^{-1}C \leqslant C\sigma^{-1}C$.

$(v) \Rightarrow (i)$ $\sigma C \leqslant \sigma C\sigma^{-1}\sigma \leqslant \sigma C\sigma^{-1}C\sigma \leqslant \sigma\sigma^{-1}C\sigma \leqslant C\sigma$. □

Remark 5. Note that, from the equivalence of (i) and (v) of the preceding lemma, it follows that C is a structural closure operator on an M-set $\langle A, \cdot\rangle$ if, and only if, for every $\sigma \in M$, the map $\sigma^{-1} : \mathcal{P}A \to \mathcal{P}A$ sends C-closed sets to C-closed sets. Hence, arbitrary intersections of structural closure systems are also structural. As a consequence, structural closure operators on an M-set $\langle A, \cdot\rangle$ form a complete sublattice of $\mathbf{Clop}(A)$.

Definition 6. Let C and D be closure operators on sets A and B, respectively. A map $f : A \to B$ is *continuous (relative to C and D)* when for every D-closed set T, $f^{-1}(T) = \{x \in A : fx \in T\}$ is a C-closed set. We define $C[A, A]$ as the set of all continuous endomaps of A relative to C.

It is obvious that, given a closure operator C on A, a composition of continuous endomaps of A relative to C is a continuous endomap. Then, it is straightforward that $\langle C[A, A], \circ, id_A\rangle$ is a monoid.

It is well-known that actions of a monoid M on a set A are in bijective correspondence with monoid homomorphisms from M to the monoid of endomaps of A. The following proposition characterizes C-compatible actions in a similar manner.

Proposition 7. *If M is a monoid, and C is a closure operator on a set A, then there is a bijective correspondence between C-compatible actions on A by M and monoid homomorphisms $M \to C[A, A]$.*

Proof. Let M be a monoid, $\langle A, \cdot\rangle$ an M-set and C a structural closure operator on $\langle A, \cdot\rangle$. Define the map: $\Psi : M \to C[A, A]$ by $\Psi(\sigma) : A \to A$ such that for every $x \in A$, $\Psi(\sigma)(x) = \sigma \cdot x$. Note that, in virtue of Remark 5, $\Psi(\sigma)$ is C-continuous, for every $\sigma \in M$, and then Ψ is well-defined. Moreover, it is obvious that $\Psi(1) = id_A$ and, if $\sigma, \sigma' \in M$ and $x \in A$, then

$$\Psi(\sigma\sigma')(x) = (\sigma\sigma') \cdot x = \sigma \cdot (\sigma' \cdot x) = \Psi(\sigma)\big(\Psi(\sigma')(x)\big) = \big(\Psi(\sigma) \circ \Psi(\sigma')\big)(x).$$

Therefore, $\Psi : M \to C[A, A]$ is a monoid homomorphism.

In order to get the converse construction, suppose that M is a monoid, C is a closure operator on a set A, and $\Psi : M \to C[A, A]$ is a monoid homomorphism. Define for every $\sigma \in M$, and $x \in A$, $\sigma \cdot x = \Psi(\sigma)(x)$. Since $\Psi(1) = id_A$, then for every $x \in A$, $1 \cdot x = x$. Moreover, for every $\sigma, \sigma' \in M$ and every $x \in A$,

$$(\sigma\sigma') \cdot x = \Psi(\sigma\sigma')(x) = \big(\Psi(\sigma) \circ \Psi(\sigma')\big)(x) = \Psi(\sigma)\big(\Psi(\sigma')(x)\big) = \sigma \cdot (\sigma' \cdot x).$$

Hence, $\cdot$ is an action.

Finally, if $X \subseteq A$, and $\sigma \in M$, $C\sigma^{-1}CX = C\Psi(\sigma)^{-1}CX = \Psi(\sigma)^{-1}CX = \sigma^{-1}CX$, since $\Psi(\sigma)$ is C-continuous. Thus, in virtue of Lemma 4, C is a structural closure operator on $\langle A, \cdot \rangle$, and hence, $\cdot$ is a C-compatible action of M on A. Clearly the correspondences we have defined are inverse of one another, and hence they are bijective. □

3.1 Examples of structural closure operators on M-sets

In what follows we present some known deductive systems as structural closure operators on M-sets. In fact, we define closure relations, instead of closure operators.

Sentential Logics Let $\mathcal{L}$ be an algebraic type, $\mathbf{Fm}_{\mathcal{L}}$ the algebra of formulas of type $\mathcal{L}$, and $M(\mathcal{L}) = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ the set of endomorphisms of $\mathbf{Fm}_{\mathcal{L}}$, which is a monoid with composition. Note that $M(\mathcal{L})$ acts on $Fm_{\mathcal{L}}$ by *evaluation*, i.e., the action is $\langle \sigma, \varphi \rangle \mapsto \sigma \cdot \varphi = \sigma(\varphi)$.

A *sentential logic* is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}}$ is a consequence relation on $Fm_{\mathcal{L}}$, i.e., a closure relation such that for every set of formulas Γ, every formula φ, and every $\sigma \in M(\mathcal{L})$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{implies} \quad \sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi.$$

That is to say, the closure operator associated with a sentential logic $\mathcal{S}$, usually denoted by $Cn_{\mathcal{S}}$, is structural on the $M(\mathcal{L})$-set $\langle Fm_{\mathcal{L}}, \cdot \rangle$. And reciprocally, every structural closure operator on $\langle Fm_{\mathcal{L}}, \cdot \rangle$ determines a sentential logic.

Consequence relations on sets of equations The set of equations of type $\mathcal{L}$ is defined as $Eq_{\mathcal{L}} = Fm^2_{\mathcal{L}}$. As usual, we denote equations $\langle \varphi, \psi \rangle$ by $\varphi \approx \psi$. It is easy to see that the product monoid $M(\mathcal{L})^2 = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}}) \times \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ acts on $Eq_{\mathcal{L}}$ in the following way: for every $\sigma = \langle \sigma_1, \sigma_2 \rangle \in M(\mathcal{L})^2$ and every equation $\varphi \approx \psi$,

$$\sigma \cdot (\varphi \approx \psi) = \sigma_1(\varphi) \approx \sigma_2(\psi).$$

If $\mathbf{A}$ is an algebra of type $\mathcal{L}$, and $h_1, h_2 : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, we say that $\langle h_1, h_2 \rangle$ *satisfies* an equation $\varphi \approx \psi$ when $h_1\varphi = h_2\psi$. And $\langle h_1, h_2 \rangle$ *satisfies* a set of equations Π when $\langle h_1, h_2 \rangle$ satisfies each of its elements. Now, for every class of algebras K of type $\mathcal{L}$, we define a closure relation $\Vdash_{\mathsf{K}}$ in virtue of this notion of satisfaction: for every set of equations Π and every equation $\varphi \approx \psi$,

$$\Pi \Vdash_{\mathsf{K}} \varphi \approx \psi \quad \Leftrightarrow \quad \forall \mathbf{A} \in \mathsf{K}, \forall h_1, h_2 : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A} \ \big(\langle h_1, h_2 \rangle \text{ satisfies } \Pi \Rightarrow \langle h_1, h_2 \rangle \text{ satisfies } \varphi \approx \psi\big).$$

It is easy to prove that $\Vdash_{\mathsf{K}}$ is $M(\mathcal{L})^2$-structural.

We can obtain a number of closure relations on $Eq_{\mathcal{L}}$ by restricting the satisfaction condition to a particular family of pairs of homomorphisms, instead of considering all of them. If $\mathcal{F}$ is a family of pairs of homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$, such that for every $h = \langle h_1, h_2 \rangle \in \mathcal{F}$, the codomain of h_1 and that of h_2 is the same $\mathcal{L}$-algebra, then we can define the closure relation $\Vdash_{\mathcal{F}}$ as follows: for every set of equations Π and every equation $\varphi \approx \psi$,

$$\Pi \Vdash_{\mathcal{F}} \varphi \approx \psi \quad \Leftrightarrow \quad \forall h \in \mathcal{F} \ (h \text{ satisfies } \Pi \Rightarrow h \text{ satisfies } \varphi \approx \psi).$$

In general, we do not have $M(\mathcal{L})^2$-structurality for these closure relations. But we always have that the set

$$\{\sigma \in M(\mathcal{L})^2 : \forall \Pi \subseteq Eq_{\mathcal{L}}, \forall \varphi \approx \psi \in Eq_{\mathcal{L}},\ \Pi \Vdash_{\mathcal{F}} \varphi \approx \psi \Rightarrow \sigma\Pi \Vdash_{\mathcal{F}} \sigma \cdot (\varphi \approx \psi)\}$$

is the universe of the bigest submonoid N of $M(\mathcal{L})^2$ such that $\Vdash_{\mathcal{F}}$ is N-structural.

The best known closure relation of satisfaction associated to a class of algebras K, called the *(semantic) equational consequence relation* determined by K and denoted[2] by $\models_{\mathsf{K}}$, is a particular example of this: for the family $\mathcal{F}_{\mathsf{K}} = \{\langle h, h\rangle : h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A},\ \mathbf{A} \in \mathsf{K}\}$ we have $\models_{\mathsf{K}} = \Vdash_{\mathcal{F}_{\mathsf{K}}}$. This closure operator is not $M(\mathcal{L})^2$-structural, but it is structural respect to the submonoid $\widetilde{M}(\mathcal{L}) = \{\langle \sigma, \sigma\rangle : \sigma \in M(\mathcal{L})\}$, which is isomorphic to the monoid $M(\mathcal{L})$. These consequence relations plays a crucial role in Abstract Algebraic Logic, since algebraizable logics are defined as those logics which are equivalent to some consequence relation of the form $\vDash_{\mathsf{K}}$. In this case, K is proved to be unique with this property and is called the *equivalent algebraic semantics of* the logic.

Another important kind of equational consequence relation that plays a role in Abstract Algebraic Logic, in particular in the theory of weakly algebraizable sentential logics, is the *surjectively structural equational consequence of* a class of algebras K. It is defined in [CJ00] (see also [Cze01]) and it is denoted by $\models^{se}_{\mathsf{K}}$. It can be obtained taking the family $\mathcal{F}^s_{\mathsf{K}}$ of those pairs $\langle h, h\rangle$ such that $h : \mathbf{Fm} \to \mathbf{A}$ is surjective and $\mathbf{A} \in \mathsf{K}$. Hence, the consequence relation $\Vdash_{\mathcal{F}^s_{\mathsf{K}}}$ is the relation $\models^{se}_{\mathsf{K}}$. It is easy to prove that, if $M^s(\mathcal{L})$ is the monoid of surjective substitutions, then $\models^{se}_{\mathsf{K}}$ is $M^s(\mathcal{L})$-structural.

Gentzen systems, m-sided Gentzen systems, and hypersequents For a fixed algebraic type $\mathcal{L}$, an $\mathcal{L}$-*sequent* is a pair of finite sequences of formulas in $Fm_{\mathcal{L}}$. If $\Gamma = \langle \gamma_1, \ldots, \gamma_n\rangle$ and $\Delta = \langle \delta_1, \ldots, \delta_m\rangle$, are finite sequences of formulas, we use $\Gamma \rhd \Delta$ to denote the sequent $\langle \Gamma, \Delta\rangle$, and define the *trace* of $\Gamma \rhd \Delta$ as $\mathrm{tr}(\Gamma \rhd \Delta) = \langle n, m\rangle \in \omega^2$. We use of the term *trace* following [Raf06]. If $Q \subseteq \omega^2$ is non-empty, then we define $Seq^Q_{\mathcal{L}}$ as the set of all $\mathcal{L}$-sequents with trace in Q.

The monoid $M(\mathcal{L}) = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ of endomorphisms of the algebra of formulas acts on $Seq^Q_{\mathcal{L}}$ in the following way: for every sequent $\Gamma \rhd \Delta = \langle \gamma_1, \ldots, \gamma_n\rangle \rhd \langle \delta_1, \ldots, \delta_m\rangle$, and every $\sigma \in M(\mathcal{L})$,

$$\sigma \cdot (\Gamma \rhd \Delta) = \big\langle \sigma(\gamma_1), \ldots, \sigma(\gamma_n)\big\rangle \rhd \big\langle \sigma(\delta_1), \ldots, \sigma(\delta_m)\big\rangle.$$

A *Gentzen relation* is a structural closure relation $\vdash_{\mathcal{G}}$ on an $M(\mathcal{L})$-set of the form $\langle Seq^Q_{\mathcal{L}}, \cdot\rangle$, where $Q \subseteq \omega^2$ is the *trace* of $\vdash_{\mathcal{G}}$. A *Gentzen system* is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}}\rangle$, where $\vdash_{\mathcal{G}}$ is a Gentzen relation on $\mathcal{L}$-sequents. Note that, our definition of Gentzen relation and Raftery's are

[2] In [Cze01] this is denoted by $\models^{eq}_{\mathsf{K}}$.

coextensive, while our definition of Gentzen system is broader than the definition in [RV95] because we do not require finitarity and we admit Gentzen systems with traces other than cartesian products $\alpha \times \beta \subseteq \omega^2$.

Similarly, an *m-sided $\mathcal{L}$-sequent* is a sequence of m finite sequences of formulas in $Fm_{\mathcal{L}}$. They are denoted thus

$$\varphi^1_1, \ldots, \varphi^1_{n_1} \mid \cdots \mid \varphi^m_1, \ldots, \varphi^m_{n_m},$$

and we define its *trace* as the m-tuple $\langle n_1, \ldots, n_m \rangle$. Given a non-empty set of traces $Q \subseteq \omega^m$, we define $Sseq^Q_{\mathcal{L}}$ the set of all m-sided $\mathcal{L}$-sequents with trace in Q. The monoid $M(\mathcal{L})$ acts in a similar way in a set of m-sided sequents: for every $\sigma \in M(\mathcal{L})$,

$$\sigma \cdot (\varphi^1_1, \ldots, \varphi^1_{n_1} \mid \cdots \mid \varphi^m_1, \ldots, \varphi^m_{n_m}) = \sigma(\varphi^1_1), \ldots, \sigma(\varphi^1_{n_1}) \mid \cdots \mid \sigma(\varphi^m_1), \ldots, \sigma(\varphi^m_{n_m}).$$

An *m-sided Gentzen system* is a pair $\langle \mathcal{L}, \vdash \rangle$ where $\vdash$ is a structural closure relation on the $M(\mathcal{L})$-set $\langle Sseq^Q_{\mathcal{L}}, \cdot \rangle$ for some set of traces $Q \subseteq \omega^m$.

An *$\mathcal{L}$-hypersequent* is a finite sequence of $\mathcal{L}$-sequents, for some language, and each element of the sequence is called a *component* of the hypersequent. We use the standard notation for hypersequents:

$$\Xi = \Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_k \rhd \Delta_k.$$

The *trace* of a hypersequent $\Xi = \Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_k \rhd \Delta_k$ is the map $T_\Xi : \{1, \ldots, k\} \to \omega^2$ such that for every i, $T_\Xi(i) = \mathrm{tr}(\Gamma_i \rhd \Delta_i)$ is the trace of the i-th component of Ξ. If Q is a non-empty set of traces of $\mathcal{L}$-hypersequents, we define the set $Hseq^Q_{\mathcal{L}}$ as the set of all $\mathcal{L}$-hypersequents with trace in Q.

The set $Hseq^Q_{\mathcal{L}}$ inherits an action of the monoid $M(\mathcal{L}) = \mathrm{End}(\mathbf{Fm}_{\mathcal{L}})$ in the following way: for every hypersequent $\Xi = \Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_k \rhd \Delta_k$ and every $\sigma \in M(\mathcal{L})$,

$$\sigma \cdot \Xi = \sigma \cdot (\Gamma_1 \rhd \Delta_1) \mid \cdots \mid \sigma \cdot (\Gamma_k \rhd \Delta_k)$$

A *hypersequent calculus* or *Gentzen system of hypersequents* is a structural closure relation on the $M(\mathcal{L})$-set $\langle Hseq^Q_{\mathcal{L}}, \cdot \rangle$, for some set of traces Q and some language $\mathcal{L}$.

Rousseau [Rou67] introduced m-sided sequents in order to build proof systems in the Gentzen style for Łukasiewicz's finitely-valued logics. They have been used also by Baaz [BFZ94] in connection with automated deduction issues, and by Gil and Rebagliato [GR00]. A classical reference for hypersequents is [Avr96].

4 Translations

The main objects presented in this section can be found by different names in literature. In [Cze01] they are called *transforms*, and in [BJ06] they are called *transformers* and the direction is reversed, i.e., a transformer from A to B is a transform from B to A. The two names are very similar, so in order to avoid confusions we prefer the name *translation* which seems more natural to us.

Definition 8. A *translation* from a set B to another set A is a map $\tau : \mathcal{P}B \to \mathcal{P}A$ such that, for every $X \subseteq B$,

$$\tau X = \bigcup_{x \in X} \tau\{x\}. \tag{2}$$

In particular, if τ is a translation, then $\tau\emptyset = \emptyset$. Note that, due to (2), in order to define a translation it is enough to define it on the singletons $\{x\}$, for $x \in B$. Translations are residuated maps. Indeed, it is straightforward to see that translations from B to A are exactly all the residuated maps from $\mathcal{P}B$ to $\mathcal{P}A$. The residuum of $\tau \in \mathrm{Trans}(B, A)$ is the map[3] $\tau^{-1} : \mathcal{P}A \to \mathcal{P}B$ defined in the following way: for every $Y \subseteq A$, $\tau^{-1}Y = \{x \in B : \tau\{x\} \subseteq Y\}$. The residuum property states that for every $X \subseteq B$ and every $Y \subseteq A$,

$$X \subseteq \tau^{-1}Y \quad \Leftrightarrow \quad \tau X \subseteq Y.$$

We define $\mathrm{Trans}(B, A)$ as the set of all translations from B to A. A translation $\tau \in \mathrm{Trans}(B, A)$ is *finitary* when for every $x \in B$, $\tau\{x\}$ is a finite set.

The following result is a reformulation of Lemma 1.

Lemma 9. *If $\tau \in \mathrm{Trans}(B, A)$, then:*

(i) τ is completely additive (i.e., it preserves all joins), and hence τ is monotone.

(ii) τ^{-1} is completely multiplicative (i.e., it preserves all meets), and hence τ^{-1} is monotone.

(iii) $id_{\mathcal{P}B} \leqslant \tau^{-1}\tau$ and $\tau\tau^{-1} \leqslant id_{\mathcal{P}A}$.

(iv) If $\rho \in \mathrm{Trans}(C, B)$, then $\tau\rho \in \mathrm{Trans}(C, A)$ and $(\tau\rho)^{-1} = \rho^{-1}\tau^{-1}$.

The following lemma will be used later:

Lemma 10. *If C is a closure operator on A, and $\tau_0, \tau_1 \in \mathrm{Trans}(B, A)$, then:*

(i) $C\tau_0 \leqslant C\tau_1$ implies $\tau_1^{-1}C \leqslant \tau_0^{-1}C$.

(ii) $C\tau_0 = C\tau_1$ implies $\tau_1^{-1}C = \tau_0^{-1}C$.

Proof. Suppose that $C\tau_0 \leqslant C\tau_1$. By using Lemma 9.(iii), we have $C\tau_0\tau_1^{-1}C \leqslant C\tau_1\tau_1^{-1}C \leqslant CC = C$. Therefore, $\tau_0\tau_1^{-1}C \leqslant C$, and thus, by Lemma 9.(iii) again, $\tau_1^{-1}C \leqslant \tau_0^{-1}\tau_0\tau_1^{-1}C \leqslant \tau_0^{-1}C$. The second statement follows from the first one. ☐

The following result has been taken by some authors as the reason to call $\tau \in \mathrm{Trans}(B, A)$ a "transformer" from A to B, since it establishes that τ "transforms" every closure operator on A into a closure operator on B.

Proposition 11. *If C is a closure operator on A and $\tau \in \mathrm{Trans}(B, A)$, then $C^\tau = \tau^{-1}C\tau$ is a closure operator on B. Furthermore, the closure system of C^τ is $\mathrm{Cl}(C^\tau) = \{\tau^{-1}T : T \in \mathrm{Cl}(C)\}$. If τ and C are finitary, then so is C^τ.*

[3]Note that τ^{-1} is not the inverse of τ, but its residuum which was denoted by τ^* in Section 2.

Proof. The set $\mathcal{D} = \{\tau^{-1}T : T \in \mathrm{Cl}(C)\} \subseteq A$ is a closure system on A, since τ^{-1} preserves arbitrary intersections. If D is the closure operator associated with $\mathcal{D}$, then for every $X \subseteq A$, we have $DX = \bigcap\{\tau^{-1}T : T \in \mathrm{Cl}(C),\ X \subseteq \tau^{-1}T\} = \tau^{-1}\bigcap\{T : T \in \mathrm{Cl}(C),\ X \subseteq \tau^{-1}T\} = \tau^{-1}\bigcap\{T : T \in \mathrm{Cl}(C),\ \tau X \subseteq T\} = \tau^{-1}C\tau T$. Thus, $D = \tau^{-1}C\tau = C^\tau$. The proof of the finitaryness of C^τ provided that C and τ are finitary is straightforward. □

Definition 12. Let C be a closure operator on A and $\tau \in \mathrm{Trans}(B, A)$. The closure operator C^τ on B is called the τ-*transformed* of C.

Now, for every closure operator D on a set B and every translation $\tau \in \mathrm{Trans}(B, A)$, we define the notion of τ-filter of D on A. We will see that they form a closure system on A.

Definition 13. Let D be a closure operator on a set B and $\tau \in \mathrm{Trans}(B, A)$. A τ-*filter* of D is a subset $T \subseteq A$ such that $\tau^{-1}T$ is a D-closed set. We define the set $\tau\text{-}\mathcal{F}il^D$ as the set of all τ-filters of D.

Since τ^{-1} is completely multiplicative, $\tau\text{-}\mathcal{F}il^D$ is a closure system on A. We denote its associated closure operator by D_τ.

The following proposition characterizes D_τ. It is well-known that, if $f : L' \to L$ is a map between lattices and for every $a \in L$ there exists $g(a) = \min\{x : a \leqslant f(x)\}$, then $g : L \to L'$ is residuated with residuum f. Thus, Corollary 15 follows immediately from it.

Proposition 14. *Let D be a closure operator on B and $\tau \in \mathrm{Trans}(B, A)$. Then D_τ is the smallest closure operator H on A such that $D \leqslant H^\tau$, that is*

$$D_\tau = \min\{H : H \in \mathrm{Clop}(A),\ D \leqslant H^\tau\}.$$

Proof. Note that, if D is a closure operator on a set B, and $\tau \in \mathrm{Trans}(B, A)$, then $\tau\text{-}\mathcal{F}il^D \subseteq \mathrm{Cl}(D)$, and hence $D \leqslant (D_\tau)^\tau$. Furthermore, if H is a closure operator on A such that $D \leqslant H^\tau$, then for every $T \in \mathrm{Cl}(H)$, $\tau^{-1}T \in \mathrm{Cl}(H^\tau) \subseteq \mathrm{Cl}(D)$, and hence $D_\tau \leqslant H$. □

Corollary 15. *If $\tau \in \mathrm{Trans}(B, A)$, then the map $(\cdot)_\tau : \mathbf{Clop}(A) \to \mathbf{Clop}(B)$ is residuated, with residuum the map $(\cdot)^\tau : \mathbf{Clop}(B) \to \mathbf{Clop}(A)$. Hence, both are monotone maps.*

5 Representations

There is a close tie between a closure operator and its τ-transformed. As we will see in this section, the ordered set of C^τ-closed sets can be embedded in the ordered set of C-closed sets. In our terms, there exists a *representation* of C^τ in C. We begin with the following crucial lemma:

Lemma 16. *If $\tau \in \mathrm{Trans}(B, A)$ and C is a closure operator on A, then*

(i) $C\tau = C\tau C^\tau$.

(ii) $\tau^{-1} \circ C\tau{\restriction}\mathrm{Cl}(C^\tau) = id_{\mathrm{Cl}(C^\tau)}$.

(iii) $C\tau \circ \tau^{-1}\restriction \mathrm{Cl}(C) \leqslant id_{\mathrm{Cl}(C)}$.

(iv) τ^{-1} *maps* $\mathrm{Cl}(C)$ *onto* $\mathrm{Cl}(C^\tau)$*, preserving all meets.*

(v) $C\tau$ *is injective from* $\mathrm{Cl}(C^\tau)$ *to* $\mathrm{Cl}(C)$ *and preserves all joins.*

Proof. (i) $C\tau C^\tau = C\tau\tau^{-1}C\tau \leqslant CC\tau = C\tau \leqslant C\tau C^\tau$.

(ii) If $X \in \mathrm{Cl}(C^\tau)$, then $\tau^{-1} \circ C\tau X = \tau^{-1}C\tau X = C^\tau X = X$.

(iii) $C\tau \circ \tau^{-1} = C\tau\tau^{-1} \leqslant C$. Thus, if $X \in \mathrm{Cl}(C)$, then $C\tau \circ \tau^{-1}X \subseteq CX = X$.

(iv) It is an immediate consequence of (ii), since we have proved that $C\tau\restriction \mathrm{Cl}(C^\tau)$ is a right inverse of $\tau^{-1}\restriction \mathrm{Cl}(C) : \mathrm{Cl}(C) \to \mathrm{Cl}(C^\tau)$. Furthermore, we saw in Lemma 9 that τ^{-1} preserves all meets.

(v) It is also an immediate consequence of (ii), since we have proved that $\tau^{-1}\restriction \mathrm{Cl}(C)$ is a left inverse of $C\tau\restriction \mathrm{Cl}(C^\tau) : \mathrm{Cl}(C^\tau) \to \mathrm{Cl}(C)$. In order to prove that $C\tau$ preserves all joins on $\mathrm{Cl}(C^\tau)$, let us suppose that $\mathcal{X} \subseteq \mathrm{Cl}(C^\tau)$. Thus, in virtue of property (1),

$$C\tau \bigvee_{X\in\mathcal{X}} X = C\tau C^\tau \bigcup_{X\in\mathcal{X}} X = C\tau \bigcup_{X\in\mathcal{X}} X = C \bigcup_{X\in\mathcal{X}} \tau X = C \bigcup_{X\in\mathcal{X}} C\tau X = \bigvee_{X\in\mathcal{X}} C\tau X. \qquad \square$$

According to the preceding lemma, the map $C\tau\restriction \mathrm{Cl}(C)^\tau : \mathrm{Cl}(C^\tau) \to \mathrm{Cl}(C)$ is an embedding of $\mathrm{Cl}(C^\tau)$ in $\mathrm{Cl}(C)$. This inspires the notion of a *representation* defined below. We want to remark the importance of this map and set a notation for it: $[C\tau] = C\tau\restriction \mathrm{Cl}(C^\tau)$.

Definition 17. Let C and D be closure operators on sets A and B, respectively. A *representation* of D in C is a map $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ such that, for all $X, X' \in \mathrm{Cl}(D)$,

$$X \subseteq X' \quad \Leftrightarrow \quad FX \subseteq FX'.$$

In particular, a representation is a morphism of orders, so we will write $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$ for a representation F of D in C. A representation is *join* when it preserves all joins.

The proof of the next characterization of join representations is straightforward.

Lemma 18. *Let C and D be closure operators on sets A and B, respectively, and* $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ *a map. Then, F is a join representation of D in C if, and only if, F preserves all joins and it is injective.* $\square$

Definition 19. A representation $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$ of D in C is *induced* by $\tau \in \mathrm{Trans}(B, A)$ when $FD = C\tau$, that is, when the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}B & \xrightarrow{\ \tau\ } & \mathcal{P}A \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle C} \\ \mathrm{Cl}(D) & \xrightarrow{\ F\ } & \mathrm{Cl}(C) \end{array}$$

We also say that D is *represented in C by τ*.

Note that, if τ induces $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$, then $C\tau D = FDD = FD = C\tau$, which is equivalent to:

$$\tau D \leqslant C\tau. \tag{3}$$

Corollary 20. *If $\tau \in \mathrm{Trans}(B, A)$, and C is a closure operator on A, then $[C\tau]$ is a join representation of C^τ in C induced by τ.*

Proof. From Lemma 16 we have that $[C\tau]$ is injective and preserves all joins. Thus, in virtue of Lemma 18, $[C\tau]$ is a join representation. Finally, from Lemma 16 we have that: $[C\tau]C^\tau = C\tau C^\tau = C\tau$, which establishes that $[C\tau]$ is induced by τ. □

Corollary 21. *If C is a closure operator and $\tau \in \mathrm{Trans}(B, A)$, then the range of $[C\tau]$ is the universe of a join-complete subsemi-lattice, $\mathbf{L}$, of the lattice $\mathbf{Cl}(C)$, and moreover $\tau^{-1}{\restriction}L$ is the inverse of $[C\tau]$. If furthermore $C\tau\tau^{-1}{\restriction}\mathrm{Cl}(C) = id_{\mathrm{Cl}(C)}$, then we have that $\mathbf{Cl}(C^\tau)$ and $\mathbf{Cl}(C)$ are isomorphic.*

Theorem 22. *Let C and D be closure operators on A and B, respectively, $\tau \in \mathrm{Trans}(B, A)$, and $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$ a representation of D in C induced by τ. Then*

(i) $D = C^\tau$,

(ii) $F = [C\tau]$.

In particular, every induced representation is join.

Proof. Suppose that $FD = C\tau$. On the one hand, $\tau D \leqslant C\tau$ by (3), and hence, $D \leqslant \tau^{-1}\tau D \leqslant \tau^{-1}C\tau = C^\tau$. On the other hand, $FDC^\tau = C\tau C^\tau = C\tau = FD$. Therefore, by the injectivity of F, $DC^\tau = D$, and hence $C^\tau \leqslant DC^\tau = D$.

Now, suppose that $X \in \mathrm{Cl}(D)$. Thus, $X = DX$, and hence $FX = FDX = C\tau X = [C\tau]X$. □

Note that, condition (i) can be expressed in the following terms: for every $X \cup \{b\} \subseteq B$,

$$X \vdash_D b \quad \Leftrightarrow \quad \tau X \vdash_C \tau\{b\}.$$

This motivates calling (*faithful, conservative*) *interpretations* the representations induced by a translation, as in logic [CJ00], and [Cze01, p. 291]. In these terms, the next theorem characterizes interpretations as join representations.

Theorem 23. *If C and D are closure operators on A and B, respectively, then $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$ is a join representation of D in C if, and only if F is induced by a translation.*

Proof. As a consequence of Theorem 22, we have that every induced representation is join. In order to prove the converse, suppose that $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ is a join representation and let τ be the translation determined by $\tau\{x\} = FD\{x\}$, for every $x \in B$. Then, in virtue of (1), if $X \subseteq B$, $FDX = FD \bigcup_{x\in X} \{x\} = FD \bigcup_{x\in X} D\{x\} = F \bigvee_{x\in X} D\{x\} = \bigvee_{x\in X} FD\{x\} = C \bigcup_{x\in X} FD\{x\} = C \bigcup_{x\in X} \tau\{x\} = C\tau X$, where each supremum is taken in the appropriate lattice. □

We are specially interested in studying representations that respect the structurality of closure operators, and in finding which translations induce these kind of representations. We define the notions of a *structural* representation and a *structural* translation. Unfortunately, a characterization for them of the sort of that in Theorem 23 is not possible, as we will see.

Definition 24. Let M be a monoid, $\langle A, \cdot\rangle$, $\langle B, \cdot\rangle$ M-sets, and C and D structural closure operators on $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$, respectively. A representation $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ of D in C is *structural* when for every $\sigma \in M$, $FD\sigma_B = C\sigma_A F$, that is, for every $\sigma \in M$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Cl}(D) & \xrightarrow{F} & \mathrm{Cl}(C) \\ {\scriptstyle D\sigma_B}\downarrow & & \downarrow{\scriptstyle C\sigma_A} \\ \mathrm{Cl}(D) & \xrightarrow{F} & \mathrm{Cl}(C) \end{array}$$

Here we write $D\sigma_B$ for $D\sigma_B {\restriction} \mathrm{Cl}(D)$, and the analogous for $C\sigma_A$. We hope this simplification of notation causes no misunderstanding.

Definition 25. Let M be a monoid, $\langle A, \cdot\rangle$ an M-set and C a structural closure operator on $\langle A, \cdot\rangle$, the structure

$$\mathbf{Cl}(C, M) = \big\langle \mathrm{Cl}(C), \langle C\sigma_A : \sigma \in M\rangle, \subseteq \big\rangle.$$

Note that for two structural closure operators C and D, a structural representation is a morphism of structures $F : \mathbf{Cl}(D, M) \to \mathbf{Cl}(C, M)$ which is, in particular, a representation of D on C. That is, representations are embeddings.

Definition 26. Let M be a monoid and $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$ M-sets. A translation $\tau \in \mathrm{Trans}(A, B)$ is *structural* when for every $\sigma \in M$, $\sigma\tau = \tau\sigma$ as set maps, i.e., $\sigma_A\tau = \tau\sigma_B$ on $\mathcal{P}B$.

The next proposition guarantees that, if τ is a structural translation, then the τ-transformed of a structural closure operator is also structural, and that the representation induced by τ is also structural. Furthermore, the closure operator of the τ-filters of a structural closure operator is also structural. Therefore, the residuated pair $\langle (\cdot)_\tau, (\cdot)^\tau\rangle$ restricts to a residuated pair on the respectives complete sublattices of structural closure operators.

Proposition 27. *Let M be a monoid, $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$ two M-sets, C and D structural closure operators on $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$, respectively, and $\tau \in \mathrm{Trans}(B, A)$ a structural translation. Then*

(i) C^τ is a structural closure operator on $\langle B, \cdot\rangle$.

(ii) D_τ is a structural closure operator on $\langle A, \cdot\rangle$.

(iii) If F is a representation of D in C induced by τ, then F is structural.

Proof. For part (i) suppose that $U \in \mathrm{Cl}(C^\tau)$ and $\sigma \in M$. Then, there exists $T \in \mathrm{Cl}(C)$ such that $U = \tau^{-1}T$, and hence $\sigma^{-1}U = \sigma^{-1}\tau^{-1}T = (\tau\sigma)^{-1}T = (\sigma\tau)^{-1}T = \tau^{-1}\sigma^{-1}T$, by the structurality of τ. Since C is structural, $\sigma^{-1}T \in \mathrm{Cl}(C)$, and hence $\sigma^{-1}U = \tau^{-1}\sigma^{-1}T \in \mathrm{Cl}(C^\tau)$.

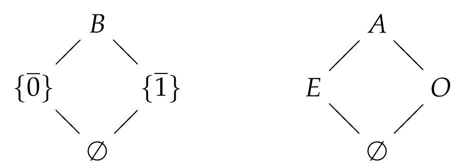
For part (ii) suppose that T is a τ-filter of D and $\sigma \in M$ is a substitution. Hence, $\tau^{-1}\sigma^{-1}T = (\sigma\tau)^{-1}T = (\tau\sigma)^{-1} = \sigma^{-1}\tau^{-1}T$, by the structurality of τ. Since T is a τ-filter of D, therefore $\tau^{-1}T \in \mathrm{Cl}(D)$, and by the structuraltiy of D, $\sigma^{-1}\tau^{-1}T \in \mathrm{Cl}(D)$. Thus, we have proved that for every τ-filter T of D and every $\sigma \in M$, $\sigma^{-1}T$ is also a τ-filter of D, that is to say, D_τ is structural.

In order to prove part (iii), suppose that F is induced by τ. Then in virtue of Theorem 22, $D = C^\tau$ and $F = [C\tau]$, and by Lemma 16.(i), $C\tau = C\tau C^\tau$, whence we obtain the equalities between the following maps on $\mathrm{Cl}(D) = \mathrm{Cl}(C^\tau)$:

$$FD\sigma = [C\tau]C^\tau\sigma = C\tau C^\tau\sigma = C\tau\sigma = C\sigma\tau = C\sigma C\tau = C\sigma[C\tau] = C\sigma F. \qquad \square$$

Nevertheless, not every structural representation is necessarily induced by a structural translation, as we see in the next example.

Let $M = \langle \mathbb{N} \setminus \{0\}, \cdot \rangle$ the monoid of nonzero natural numbers with the usual product, $A = \mathbb{N}$ the set of natural numbers and $B = \mathbb{Z}/(2)$ the set of classes of integers modulo 2. The monoid M acts on A and B by multiplication: for every $\sigma \in M$, $a \in A$, and $\overline{b} \in B$, $\sigma \cdot a$ is the product of σ and a, and $\sigma \cdot \overline{b} = \overline{\sigma b}$. Let C and D be the closure operators on A and B, respectively, with the following lattices of closed sets:



where E is the set of even numbers and O is the set of odd numbers. We prove that C and D are structural closure operators on $\langle A, \cdot \rangle$ and $\langle B, \cdot \rangle$, respectively. Indeed, as for D, since all subsets of B are D-closed, then $D = id_{\mathcal{P}B}$, and thus the structurality is evident. As for C, suppose that $\sigma \in M$ and $X \subseteq A$. If $X = \emptyset$, then it is obvious that $C\sigma C\emptyset = \emptyset = C\sigma\emptyset$. If $X \neq \emptyset$ and σ is even, then both σCX and σX are nonemtpy and subsets of E, and hence $C\sigma CX = E = C\sigma X$. If $X \neq \emptyset$ and σ is odd, then we distinguish three cases:

(i) if $X \subseteq E$, then $\sigma CX = \sigma E \subseteq E$, and $\sigma X \subseteq E$, and hence $C\sigma CX = E = C\sigma X$;

(ii) if $X \subseteq O$, then $\sigma CX = \sigma O \subseteq O$, and $\sigma X \subseteq O$, and then $C\sigma CX = O = C\sigma X$;

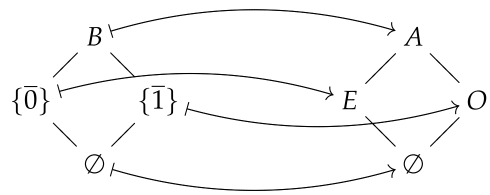
(iii) otherwise X has evens and odds, and in this case both $\sigma CX = \sigma\mathbb{N}$ and σX have evens and odds, whence $C\sigma CX = \mathbb{N} = C\sigma X$.

We prove now that D is not the τ-transformed of C by any structural translation τ. If $\tau \in \mathrm{Trans}(B, A)$ is structural, then the following diagrams would commute:

$$\begin{array}{ccc} \mathcal{P}B & \xrightarrow{\tau} & \mathcal{P}A \\ {\scriptstyle 2\cdot}\downarrow & & \downarrow{\scriptstyle 2\cdot} \\ \mathcal{P}B & \xrightarrow{\tau} & \mathcal{P}A \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P}B & \xrightarrow{\tau} & \mathcal{P}A \\ {\scriptstyle 3\cdot}\downarrow & & \downarrow{\scriptstyle 3\cdot} \\ \mathcal{P}B & \xrightarrow{\tau} & \mathcal{P}A \end{array}$$

and then $3 \cdot \tau\{\bar{1}\} = \tau\{3 \cdot \bar{1}\} = \tau\{\bar{1}\}$. In particular, $\tau\{\bar{1}\} \subseteq \mathcal{P}A$ would be invariant under the product by 3. The only subsets of $A = \mathbb{N}$ with this property are $\emptyset$ and $\{0\}$. Furthermore, $\tau\{\bar{0}\} = \tau\, 2 \cdot \{\bar{1}\} = 2 \cdot \tau\{\bar{1}\}$. In any case, $\tau\{\bar{0}\} = \tau\{\bar{1}\}$, whence $\tau^{-1}C\tau\{\bar{0}\} = \tau^{-1}C\tau\{\bar{1}\}$, and thus $D \neq C^{\tau}$.

Consider now the isomorphism $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$ determined by $F\{\bar{0}\} = E$, that is:



The proof that F is structural is a simple computation. First, $FD\sigma\emptyset = \emptyset = C\sigma F\emptyset$, and $FD\sigma\{\bar{0}\} = E = C\sigma F\{\bar{0}\}$, for arbitrary $\sigma \in M$. Furthermore, if $\sigma \in M$ is even, then $FD\sigma\{\bar{1}\} = E = C\sigma F\{\bar{1}\}$, and $FD\sigma B = E = C\sigma FB$; and if σ is odd, then $FD\sigma\{\bar{1}\} = O = C\sigma F\{\bar{1}\}$, and $FD\sigma B = A = C\sigma FB$. Thus, F is a structural representation of D in C (actually, an isomorphic one). Since we have shown that D is not the τ-transformed of C by any structural translation, F is not induced by any structural translation.

In the next sections we find a sufficient condition on M-sets which ensures that every structural representation of a structural closure operator on an M-set with this property is induced by a structural translation.

6 Graduations and graded variables

Blok and Jónsson introduced in [BJ06] the notions of a *basis* of an M-set and a *regular* M-set, which is an M-set having a basis. The notion of a basis tries to capture the role of the variables as set of generators of the free algebra of fomulas $\mathbf{Fm}_{\mathcal{L}}$. In particular that, for every formula φ, there exists a substitution (which is unique) that sends every variable to φ.

Maybe the most elementary property of a variable is that it is an element that can be sent to any other element whatsoever by a suitable substitution. In some deductive systems such as Gentzen systems there is no element with this property, because subtitutions do not change the trace of a sequent. But, for each trace there exist elements that play a similar role among all sequents with the same trace. This motivates the notions of a graded M-set and a graded variable of a graded M-set, which roughly speaking is a family of variables, one of each degree.

A graduation of an M-set $\langle A, \cdot \rangle$ is a partition $\{A_i : i \in I\}$ of the set A in parts that are stable by the action of M. That is, for every part A_i, and for every $\sigma \in M$, $\sigma A_i \subseteq A_i$. All the elements in a part will be said to have the same *degree*. For technical reasons, we prefer to formalize this concept as follows:

Definition 28. Let M be a monoid. A *graduation* on an M-set $\langle A, \cdot \rangle$ is an exhaustive map $\imath : A \to I$ such that for every $a \in A$, and every $\sigma \in M$, $\imath(\sigma a) = \imath(a)$. A *graded* M-set is an M-set equipped with a graduation. Graded M-sets will be represented as $\langle A, \cdot, \imath \rangle$ if there is no confusion about the set of *degrees* I.

If $\langle A, \cdot, \imath \rangle$ is a graded M-set, for every degree i and every $X \subseteq A$, we denote with $X_i = \{x \in X : \imath(x) = i\}$ the set of elements of X of degree i. Note that in particular $\{A_i : i \in I\}$ is a partition of A, and that for any $a, b \in A$ of different degree there is no $\sigma \in M$ such that $\sigma a = b$. In other words, for every $i \in I$ and $\sigma \in M$, $\sigma A_i \subseteq A_i$ and $\sigma^{-1} A_i \subseteq A_i$.

For every M-set $\langle A, \cdot \rangle$ there is a *trivial* graduation with exactly one degree, $\imath : A \to \{0\}$. We can extend the Blok-Jónsson notion of a basis for an M-set to that of a basis for a graded M-set, in such a way that bases for an M-set $\langle A, \cdot \rangle$ coincide with bases for the graded M-set $\langle A, \cdot, \imath \rangle$ where $\imath$ is the trivial graduation. A *basis* for a graded M-set $\langle A, \cdot, \imath \rangle$ is a set $P \subseteq A$ such that, for every $i \in I$, $P_i \neq \emptyset$, and

$$\forall a \in A, \exists! \sigma \in M, \forall p \in P_{\imath(a)},\ \sigma p = a. \tag{4}$$

If P is a basis, then for every $a \in A$ let κ^a be the unique element of M such that $\kappa^a P_{\imath(a)} = \{a\}$, that is, κ^a sends every $p \in P_{\imath(a)}$ to a.

Note that, if $\langle A, \cdot, \imath \rangle$ is a graded M-set and P is a basis for $\langle A, \cdot, \imath \rangle$, then for every $\sigma \in M$ and every $a \in A$ we have $(\sigma \kappa^a) P_{\imath(\sigma a)} = \sigma\big(\kappa^a P_{\imath(\sigma a)}\big) = \sigma\big(\kappa^a P_{\imath(a)}\big) = \sigma\{a\} = \{\sigma a\}$. And then, by the uniqueness condition in the definition of $\kappa^{\sigma a}$, we obtain

$$\sigma \kappa^a = \kappa^{\sigma a}. \tag{5}$$

This motivates the following definitions:

Definition 29. Let M be a monoid and $\langle A, \cdot \rangle$ an M-set. A *coherent family of substitutions* is a map $\kappa : A \to M$ such that for every $a \in A$ and every $\sigma \in M$, $\sigma \kappa_a = \kappa_{\sigma a}$. Here, we use κ_a as a notation for $\kappa(a)$.

Definition 30. Let M be a monoid and $\langle A, \cdot, \imath \rangle$ a graded M-set. A *graded variable* for $\langle A, \cdot, \imath \rangle$ is a map $p : I \to A$ such that for every $i \in I$, $\imath(p_i) = i$, and there exists a coherent family of substitutions κ, such that for every $a \in A$, $\kappa_a p_{\imath(a)} = a$. Here, we use p_i as a notation for $p(i)$.

Remark. Note that, if M is a monoid and $\langle A, \cdot, \imath \rangle$ is a graded M-set with a basis P, and $\kappa : A \to M$ is the map such that for every $a \in A$, $\kappa_a = \kappa^a$ the unique element of M with the property that $\kappa^a P_{\imath(a)} = \{a\}$, then every map $p : I \to P$ such that $p_i \in P_i$ is a graded variable, since in virtue of (5), κ is a coherent family of substitutions and for every $a \in A$, $\kappa_a p_{\imath(a)} = \kappa^a p_{\imath(a)} \in \kappa^a P_{\imath(a)} = \{a\}$, that is $\kappa_a p_{\imath(a)} = a$. Hence, every graded M-set having a basis has a graded variable. The notion of a graded variable is however weaker (see below the example of Gentzen systems), but enough for our purposes.

6.1 Examples of graded M-sets with a graded variable

Graded variables in $M(\mathcal{L})$-sets for formulas and equations As was proved in [BJ06], the $M(\mathcal{L})$-sets of formulas $Fm_{\mathcal{L}}$ and of equations $Eq_{\mathcal{L}}$ are *regular*, that is to say, they have bases. The set of sentential variables V is the unique basis of $Fm_{\mathcal{L}}$, and for every partition of V in two non-empty sets, $\{S, S'\}$, the set $P = \{s \approx s' : s \in S,\ s' \in S'\}$ is a basis for $Eq_{\mathcal{L}}$. Therefore, they have bases as graded $M(\mathcal{L})$-sets with the trivial graduations. As a consequence, they have graded variables. In the case of the set of formulas, every sentential variable is a graded variable, and in the case of the set of equations, for every pair of different sentential variables s and s', the equation $s \approx s'$ is a graded variable. Here, the adjective "graded" could be suppressed, since in both cases the graduation involved is the trivial one, which has only one degree.

Gentzen systems as graded $M(\mathcal{L})$-sets with a graded variable Recall that, if $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ and $\Delta = \langle \delta_1, \ldots, \delta_m \rangle$ are sequences of $\mathcal{L}$-formulas, then the trace of the $\mathcal{L}$-sequent $\Gamma \rhd \Delta$ is $\mathrm{tr}(\Gamma \rhd \Delta) = \langle n, m \rangle$. It is easy to show that $\langle Seq^{Q}_{\mathcal{L}}, \cdot, \mathrm{tr} \rangle$ is a graded $M(\mathcal{L})$-set, where the graduation is the trace map $\mathrm{tr} : Seq^{Q}_{\mathcal{L}} \to Q$.

Next we show that most of the Gentzen systems have graded variables. Only those with the sequent $\emptyset \rhd \emptyset$ and without any constant in their language do not have any graded variable. (It can be shown furthermore that Gentzen systems without the sequent $\emptyset \rhd \emptyset$ have bases.)

Proposition 31. *Let $Q \subseteq \omega$ be a non-empty set. Then, $\langle Seq^{Q}_{\mathcal{L}}, \cdot \rangle$ has a graded variable if, and only if, $\mathcal{L}$ has a constant or $\langle 0, 0 \rangle \notin Q$.*

Proof. Let $V = \{x_i : i \geqslant 1\}$ be an enumeration of the sentential variables, and for every $\langle n, m \rangle \in \omega^2$, let $p_{\langle n,m \rangle} = \langle x_1, \ldots, x_n \rangle \rhd \langle x_{n+1}, \ldots, x_{n+m} \rangle$. In particular, $p_{\langle 0,0 \rangle} = \emptyset \rhd \emptyset$.

We define for every sequent $\Gamma \rhd \Delta = \langle \eta_1, \ldots, \eta_n \rangle \rhd \langle \eta_{n+1}, \ldots, \eta_{n+m} \rangle$ of trace in $Q \setminus \{\langle 0, 0 \rangle\}$, the substitution $\kappa_{\Gamma \rhd \Delta} \in M(\mathcal{L})$ as the unique one such that

(i) $\kappa_{\Gamma \rhd \Delta}(x_i) = \eta_i$, for $1 \leqslant i \leqslant n + m$,

(ii) $\kappa_{\Gamma \rhd \Delta}(x_i) = \eta_{n+m}$, for $i > n + m$.

If $\langle 0, 0 \rangle \in Q$ and $c \in Fm_{\mathcal{L}}$ is a constant, then we define $\kappa_{\emptyset \rhd \emptyset}(x) = c$, for every $x \in V$.

Observe that, if $\Gamma \rhd \Delta$ is a sequent of trace $\langle n, m \rangle \neq \langle 0, 0 \rangle$, and $\sigma \in M(\mathcal{L})$, then $\sigma\kappa_{\Gamma \rhd \Delta}(x_i) = \sigma(\eta_i) = \kappa_{\sigma \cdot (\Gamma \rhd \Delta)}(x_i)$, if $1 \leqslant i \leqslant n + m$, and $\sigma\kappa_{\Gamma \rhd \Delta}(x_i) = \sigma(\eta_{n+m}) = \kappa_{\sigma \cdot (\Gamma \rhd \Delta)}(x_i)$, if $i > n + m$. Furthermore, $\sigma\kappa_{\emptyset \rhd \emptyset}(x) = \sigma(c) = c = \kappa_{\emptyset \rhd \emptyset}(x) = \kappa_{\sigma \cdot (\emptyset \rhd \emptyset)}(x)$, for every $x \in V$.

Thus, $\kappa = \{\kappa_{\Gamma \rhd \Delta} : \Gamma \rhd \Delta \in Seq^{Q}_{\mathcal{L}}\}$ is a coherent family of substitutions which renders the map $p : Q \to Seq^{Q}_{\mathcal{L}}$ a graded variable.

Finally, if $Fm_{\mathcal{L}}$ has no constant and $\emptyset \rhd \emptyset$ is in $\mathcal{G}$, then we show that $\mathcal{G}$ has no graded variable. In order to get a contradiction, suppose that p was a graded variable and κ a coherent family of substitutions for p. Since $\emptyset \rhd \emptyset$ is the unique sequent of trace $\langle 0, 0 \rangle$, we have $p_{\langle 0,0 \rangle} = \emptyset \rhd \emptyset$. Let x be a sentential variable, $\varphi = \kappa_{p_{\langle 0,0 \rangle}}(x)$ which is not a constant by hypothesis, and let $\sigma \in M(\mathcal{L})$ be a substitution such that $\sigma\varphi \neq \varphi$. Thus, $\varphi = \kappa_{p_{\langle 0,0 \rangle}}(x) = \kappa_{\emptyset \rhd \emptyset}(x) = \kappa_{\sigma \cdot (\emptyset \rhd \emptyset)}(x) = \sigma\kappa_{\emptyset \rhd \emptyset}(x) = \sigma\varphi$ in contradiction with the choice of σ. ☐

Remark. Note that, although a Gentzen system with the sequent $\emptyset \rhd \emptyset$ has a graded variable provided that there exists some constant in its language, it never has a graded basis. The obstacle arises from the fact that if P is a set of sequents with a sequent of every trace, then $P_{\langle 0,0 \rangle} = \{\emptyset \rhd \emptyset\}$, and thus the uniqueness of condition (4) fails.

Hypersequent systems It can be easily proved that $\langle Hseq^{Q}_{\mathcal{L}}, \cdot, T\rangle$ is a graded $M(\mathcal{L})$-set, where Q is a non-empty set of traces of hypersequents, and $T : Hseq^{Q}_{\mathcal{L}} \to Q$ is the map such that for every hypersequent Ξ, $T(\Xi) = T_{\Xi}$ is the trace of Ξ.

For every $k \geqslant 0$ there is a *k-empty* hypersequent, $\mathbf{\Phi}_k$, which is determined by the *k-empty* trace $t : \{1, \cdots, k\} \to \omega^2$ such that for every i, $t(i) = \langle 0, 0\rangle$. In particular, the 0-empty hypersequent is the empty sequence, $\mathbf{\Phi}_0 = \emptyset$.

Proposition 32. *Let Q be a non-empty set of traces of hypersequents. Then, $\langle Hseq^{Q}_{\mathcal{L}}, \cdot\rangle$ has a graded variable provided that it does not contain any k-empty sequent.*

Proof. Suppose that $Hseq^{Q}_{\mathcal{L}}$ does not contain any k-emtpy sequence. For every trace $t : \{1, \dots, k\} \to \omega^2$ in Q, and every $i \in \{1, \dots, k\}$, let $\langle n_i, m_i\rangle = t(i)$, and $X_t = \{x^i_r : 1 \leqslant i \leqslant k,\ 1 \leqslant r \leqslant n_i + m_i\}$ be a set of $\sum_{i=1}^{k}(n_i + m_i)$ different variables. Let p_t be the hypersequent

$$p_t = x^1_1, \dots, x^1_{n_1} \rhd x^1_{n_1+1}, \dots, x^1_{n_1+m_1} \mid \cdots \mid x^k_1, \dots, x^k_{n_k} \rhd x^k_{n_k+1}, \dots, x^k_{n_k+m_k},$$

and let z be a fixed sentential variable in p_t, for instance the "last one" in the evident order, which exists since t is not the k-empty trace, and for every hypersequent $\Xi = \Gamma_1 \rhd \Delta_1 \mid \cdots \mid \Gamma_k \rhd \Delta_k$ of trace $T_{\Xi} = t$, where $\Gamma_i \rhd \Delta_i = \langle \eta^i_1, \dots, \eta^i_{n_i}\rangle \rhd \langle \eta^i_{n_i+1}, \dots, \eta^i_{n_i+m_i}\rangle$, let $\kappa_{\Xi} \in M(\mathcal{L})$ be the substitution such that

(i) $\kappa_{\Xi}(x^i_j) = \eta^i_j$, for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n_i + m_i$,

(ii) $\kappa_{\Xi}(v) = \kappa_{\Xi}(z)$, for every sentential variable $v \notin X_t$.

Therefore, $p : Q \to Hseq^{Q}_{\mathcal{L}}$ is a graded variable and $\kappa = \{\kappa_{\Xi} : \Xi \in Hseq^{Q}_{\mathcal{L}}\}$ is a coherent family of substitutions for p. □

7 Structural representations and graded M-sets

The main result in this section is Theorem 36, an analog to Theorem 23 for structural representations. We saw that, in general, structural representations need not be induced by structural translations. But in the presence of graded variables, they are. The proof is divided in two lemmas (34 and 35) which are interesting in themselves and will be used in another proof. Lemma 34 characterizes structural translations from a graded set having a graded variable. In order to motivate it, let us first see how structural translations between Gentzen systems are. We need the following result:

Lemma 33. *Let M be a monoid, $\langle A, \cdot\rangle$ an M-set, $\langle B, \cdot, \imath\rangle$ a graded M-set, p a graded variable for $\langle B, \cdot, \imath\rangle$, κ a coherent family of substitutions for p, and $\tau \in \mathrm{Trans}(B, A)$ a structural translation. Then for every degree i, every element of $\tau\{p_i\}$ is invariant under κ_{p_i}.*

Proof. For every $i \in I$,

$$\kappa_{p_i}\tau\{p_i\} = \tau\kappa_{p_i}\{p_i\} = \tau\{\kappa_{p_i}p_i\} = \tau\{p_i\}.$$

Thus, if $y \in \tau\{p_i\}$, there is some $y' \in \tau\{p_i\}$ such that $y = \kappa_{p_i} y'$. Therefore,

$$\kappa_{p_i} y = \kappa_{p_i} \kappa_{p_i} y' = \kappa_{p_i} y' = y. \qquad \square$$

Let $Q_1, Q_2 \subseteq \omega^2$ be two non-empty sets such that $\langle 0,0 \rangle \notin Q_i$, $i = 1,2$, and suppose that $\tau \in \mathrm{Trans}(Seq_{\mathcal{L}}^{Q_1}, Seq_{\mathcal{L}}^{Q_2})$ is a structural translation. Let p and κ be the graded variable and the coherent family of substitutions defined in the proof of Proposition 31 for $\langle Seq_{\mathcal{L}}^{Q_1}, \cdot, \mathrm{tr} \rangle$ and set $\Pi^{\langle n,m \rangle} = \tau\{p_{\langle n,m \rangle}\} \subseteq Seq_{\mathcal{L}}^{Q_2}$. Thus, in virtue of Lemma 33 we have that every element of $\Pi^{\langle n,m \rangle}$ is invariant under the substitution $\kappa_{p_{\langle n,m \rangle}}$. Hence, $\Pi^{\langle n,m \rangle}$ is a set of $\mathcal{L}$-sequents with trace in Q_2 in the sentential variables $\{x_1, \ldots, x_{n+m}\}$, and for every $\Gamma \rhd \Delta \in Seq_{\mathcal{L}}^{Q_1}$ of trace $\langle n, m \rangle$,

$$\begin{aligned} \tau\{\Gamma \rhd \Delta\} &= \tau \kappa_{\Gamma \rhd \Delta}\{p_{\langle n,m \rangle}\} = \kappa_{\Gamma \rhd \Delta} \tau\{p_{\langle n,m \rangle}\} = \kappa_{\Gamma \rhd \Delta} \Pi^{\langle n,m \rangle} \\ &= \Pi^{\langle n,m \rangle}[\Gamma / \{x_1, \ldots, x_n\}, \Delta / \{x_{n+1}, \ldots, x_{n+m}\}], \end{aligned}$$

where $\Pi^{\langle n,m \rangle}[\Gamma / \{x_1, \ldots, x_n\}, \Delta / \{x_{n+1}, \ldots, x_{n+m}\}]$ is the set of $\mathcal{L}$-sequents resulting from replacing the variables in $\langle x_1, \ldots, x_{n+m} \rangle$ by the formulas in Γ, Δ, in every $\mathcal{L}$-sequent of $\Pi^{\langle n,m \rangle}$. Thus, structural translations from $\langle Seq_{\mathcal{L}}^{Q_1}, \cdot \rangle$ to $\langle Seq_{\mathcal{L}}^{Q_2}, \cdot \rangle$ are exactly (Q_1, Q_2)-translations in the sense of [RV95].

Lemma 34. *Let $\langle A, \cdot \rangle$ be an M-set, $\langle B, \cdot, \imath \rangle$ a graded M-set, p a graded variable for $\langle B, \cdot, \imath \rangle$, and κ a coherent family of substitutions for p. Then for every map $S : I \to \mathcal{P}A$, there exists a unique structural translation $\tau \in \mathrm{Trans}(B, A)$ determined by the condition:*

$$\forall i \in I,\ \tau\{p_i\} = \kappa_{p_i} S(i).$$

Proof. Let $\tau \in \mathrm{Trans}(B, A)$ be the translation such that, for every $b \in B$,

$$\tau\{b\} = \kappa_b S(\imath(b)).$$

Obviously, τ satisfies the required condition, since for every $i \in I$, $\imath(p_i) = i$. In order to verify the structurality of τ, suppose that $\sigma \in M$ and $b \in B$ are arbitrary elements. Therefore,

$$\tau\sigma\{b\} = \tau\{\sigma b\} = \kappa_{\sigma b} S(\imath(\sigma b)) = \sigma \kappa_b S(\imath(b)) = \sigma\tau\{b\}.$$

We only have to prove the uniqueness of τ. Let $\tau' \in \mathrm{Trans}(B, A)$ be a structural translation such that for every $i \in I$, $\tau'\{p_i\} = \kappa_{p_i} S(i)$. If $b \in B$ and $i = \imath(b)$, then

$$\tau'\{b\} = \tau'\{\kappa_b p_i\} = \tau' \kappa_b \{p_i\} = \kappa_b \tau'\{p_i\} = \kappa_b \kappa_{p_i} S(i) = \kappa_{\kappa_b p_i} S(i) = \kappa_b S(i).$$

That is, $\tau' = \tau$. $\square$

Lemma 35. *Let M be a monoid, $\langle A, \cdot \rangle$ an M-set, $\langle B, \cdot, \imath \rangle$ a graded M-set with a graded variable p, C and D structural closure operators on $\langle A, \cdot \rangle$ and $\langle B, \cdot \rangle$, respectively, and $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ a structural join representation of D in C. If for every $i \in I$, $FD\{p_i\} = C\tau\{p_i\}$, then F is induced by τ .*

Proof. If $X \subseteq B$, then $DX = \bigvee_{x\in X} D\{x\}$, and hence, $FDX = F \bigvee_{x\in X} D\{x\} = \bigvee_{x\in X} FD\{x\}$. We have also that $C\tau X = C \bigcup_{x\in X} \tau\{x\} = C \bigcup_{x\in X} C\tau\{x\} = \bigvee_{x\in X} C\tau\{x\}$.

Thus, we only need to prove that for every $b \in B$, $FD\{b\} = C\tau\{b\}$. Let κ be a coherent family of substitutions for p. If $b \in B$ and $i = \imath(b)$, then

$$\begin{aligned} FD\{b\} &= FD\{\kappa_b p_i\} = FD\kappa_b\{p_i\} = FD\kappa_b D\{p_i\} = C\kappa_b FD\{p_i\} \\ &= C\kappa_b C\tau\{p_i\} = C\kappa_b \tau\{p_i\} = C\tau\kappa_b\{p_i\} = C\tau\{\kappa_b p_i\} = C\tau\{b\}. \end{aligned}$$ □

Theorem 36. *Let M be a monoid, $\langle A, \cdot\rangle$ an M-set, $\langle B, \cdot, \imath\rangle$ a graded M-set with a graded variable p, and C and D structural closure operators on $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$, respectively. Then every structural join representation of D in C is induced by a structural translation. If moreover D is finitary, then τ can be taken to be also finitary.*

Proof. In the hypothesis of the theorem, suppose that $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ is a structural join representation of D in C. Let κ be a coherent family of substitutions for p and $S : I \to \mathcal{P}(A)$ be the map defined by $S(i) = FD\{p_i\}$. Thus, by Lemma 34, there exists a unique structural translation τ such that for every $i \in I$, $\tau\{p_i\} = \kappa_{p_i} FD\{p_i\}$. Therefore, we have that, for every $i \in I$,

$$FD\{p_i\} = FD\kappa_{p_i}\{p_i\} = FD\kappa_{p_i} D\{p_i\} = C\kappa_{p_i} FD\{p_i\} = C\tau\{p_i\}.$$

And finally, by Lemma 35, we have that F is induced by τ.

Let us suppose that D is finitary. We prove that τ can be replaced by a finitary translation τ'. Fist, we prove that for every $b \in B$, the set $D\{b\}$ is compact: If $\{X_l : l \in \Lambda\} \subseteq \mathrm{Cl}(D)$ is a family of D-closed sets, then

$$D\{b\} \subseteq \bigvee_{l\in\Lambda} X_l \quad \Rightarrow \quad b \in \bigvee_{l\in\Lambda} X_l = D \bigcup_{l\in\Lambda} X_l.$$

By finitarity of D, there is a finite set $X \subseteq \bigcup_{l\in\Lambda} X_l$ such that $b \in DX$. Since X is finite, there is a finite set $\Lambda' \subseteq \Lambda$ such that $X \subseteq \bigcup_{l\in\Lambda'} X_l$, and hence, $b \in DX \subseteq D\bigcup_{l\in\Lambda'} X_l = \bigvee_{l\in\Lambda'} X_l$.

Since F is a join representation, for every $b \in B$, $C\tau\{b\} = FD\{b\}$ is also compact. Note that, for every $b \in B$, $C\tau\{b\} = \bigvee_{x\in\tau\{b\}} C\{x\}$, and by compactness of $C\tau\{b\}$ and monotonicity of C, there is a finite subset $S \subseteq \tau\{b\}$ such that $C\tau\{b\} = \bigvee_{x\in S} C\{x\}$. In particular, for every $i \in I$, there is a finite subset $S(i) \subseteq \tau\{p_i\}$ such that $CS(i) = C\tau\{p_i\}$.

In virtue of Lemma 33, every element of $\tau\{p_i\}$ is invariant under κ_{p_i}, and hence, for every $i \in I$, $\kappa_{p_i} S(i) = S(i)$. By Lemma 34, the translation $\tau' \in \mathrm{Trans}(B, A)$ determined by

$$\tau'\{b\} = \kappa_b S(\imath(b))$$

is structural, and since $S(i)$ is finite for every $i \in I$, it is also finitary. Now, we have that, for every $i \in I$,

$$C\tau'\{p_i\} = C\kappa_{p_i} S(i) = CS(i) = C\tau\{p_i\} = FD\{p_i\}.$$

Hence, by Lemma 35, τ' induces F. □

8 Applications to similarities and equivalences

In this section we apply our previous results to the more restricted case of similarities and equivalences of closure operators and structural closure operators.

Definition 37. Let C and D be closure operators on A and B, respectively. Two translations $\tau \in \mathrm{Trans}(B, A)$ and $\rho \in \mathrm{Trans}(A, B)$ are *mutually inverse* (with respect to C and D) if, and only if, for every $a \in A$ and $b \in B$,

(i) $a \dashv\vdash_D \tau\rho\{a\}$,

(ii) $b \dashv\vdash_C \rho\tau\{b\}$.

Using that τ and ρ are translations, it is easy to see that $(i) \Leftrightarrow D = D\rho\tau$ and, analogously, $(ii) \Leftrightarrow C = C\tau\rho$.

Definition 38. Let C and D be closure operators on A and B respectively. A *similarity* between C and D is a lattice isomorphism $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$. The closure operators C and D are *similar*, in symbols $C \sim D$, if and only if there exists a similarity between C and D.

Note that a similarity between two closure operators C and D is just a bijective representation $F : \mathbf{Cl}(D) \to \mathbf{Cl}(C)$. It follows that $F^{-1} : \mathbf{Cl}(C) \to \mathbf{Cl}(D)$ is also a representation.

Definition 39. Let C and D be closure operators on A and B respectively, F a similarity between C and D, and $\tau \in \mathrm{Trans}(B, A)$ and $\rho \in \mathrm{Trans}(A, B)$. Then τ, ρ *induce* F if, and only if, τ induces F and ρ induces F^{-1} as representations.

This is equivalent to the commutativity of the two following diagrams:

$$\begin{array}{ccc} \mathcal{P}B & \xrightarrow{\ \tau\ } & \mathcal{P}A \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle C} \\ \mathrm{Cl}(D) & \xrightarrow{\ F\ } & \mathrm{Cl}(C) \end{array} \qquad \begin{array}{ccc} \mathcal{P}B & \xleftarrow{\ \rho\ } & \mathcal{P}A \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle C} \\ \mathrm{Cl}(D) & \xleftarrow{\ F^{-1}\ } & \mathrm{Cl}(C) \end{array} \tag{6}$$

Lemma 40. *If C and D are closure operators on A and B respectively, $\tau \in \mathrm{Trans}(B, A)$ and $\rho \in \mathrm{Trans}(A, B)$, then τ, ρ induce a similarity between C and D if, and only if,*

(i) $C\tau{\restriction}\mathrm{Cl}(D)$ and $D\rho{\restriction}\mathrm{Cl}(C)$ are lattice homomorphisms inverse of one another, and

(ii) $C\tau = C\tau D$ and $D\rho = D\rho C$.

Proof. If τ and ρ induce a similarity between C and D, then there exists an isomorphism $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ such that the diagrams (6) commute. Thus, F and F^{-1} are representations induced by τ and ρ, respectively. In virtue of Theorem 22, $D = C^{\tau}$, $C = D^{\rho}$, $F = [C\tau] = C\tau{\restriction}\mathrm{Cl}(D)$ and $F^{-1} = [D\rho] = D\rho{\restriction}\mathrm{Cl}(C)$, which proves (i). Furthermore, (ii) follows from (3).

In order to prove the other implication, note that (i) estates that $F = C\tau\restriction \mathrm{Cl}(D)$ is a lattice isomorphism between $\mathrm{Cl}(D)$ and $\mathrm{Cl}(C)$ with inverse $F^{-1} = D\rho\restriction \mathrm{Cl}(C)$ and, moreover, $FD = C\tau D = C\tau$, and $F^{-1}C = D\rho C = D\rho$ by (ii). Thus, τ and ρ induce a similarity between C and D. □

Using Theorem 22 and Corollary 21 one readily proves:

Proposition 41. *Let C and D be closure operators on A and B respectively, and $\tau \in \mathrm{Trans}(B, A)$ and $\rho \in \mathrm{Trans}(A, B)$ be such that they induce a similarity between C and D. Then:*

(i) $D = C^\tau$ *and* $C = D^\rho$.

(ii) $C\tau X = \rho^{-1}X$, *for all* $X \in \mathrm{Cl}(D)$.

(iii) $D\rho Y = \tau^{-1}Y$, *for all* $Y \in \mathrm{Cl}(C)$.

(iv) *τ and ρ are mutually inverse, with respect to C and D.*

With this, we can obtain an alternative proof of Blok-Jónsson theorem:

Theorem 42. *If C and D are closure operators on A and B respectively, $\tau \in \mathrm{Trans}(B, A)$, and $\rho \in \mathrm{Trans}(A, B)$, then the following conditions are equivalent:*

(i) *τ, ρ induce a similarity between C and D.*

(ii) $D = C^\tau$ *and* $C\tau\rho = C$.

(iii) $C = D^\rho$ *and* $D\rho\tau = D$.

Proof. First we prove that $(ii) \Leftrightarrow (iii)$, and after this we will prove $(i) \Leftrightarrow (ii)$.

Suppose that $D = C^\tau$ and $C\tau\rho = C$. Note that $id_{\mathcal{P}A}$ and $\tau\rho$ are both translations from A to A. Given that $C\tau\rho = C = Cid_{\mathcal{P}A}$, applying Lemma 10, we obtain that $C = id_{\mathcal{P}A}^{-1}C = (\tau\rho)^{-1}C = \rho^{-1}\tau^{-1}C$. Hence, we have $D^\rho = \rho^{-1}D\rho = \rho^{-1}C^\tau\rho = \rho^{-1}\tau^{-1}C\tau\rho = C\tau\rho = C$. We also have that $D\rho\tau = C^\tau\rho\tau = \tau^{-1}C\tau\rho\tau = \tau^{-1}C\tau = C^\tau = D$. Hence we have proved $(ii) \Rightarrow (iii)$, and by symmetry the converse follows.

In the previous proposition we have proved $(i) \Rightarrow (ii)$. Suppose now (ii), that is to say, $D = C^\tau$ and $C\tau\rho = C$. Thus, we have:

$$D\rho C\tau D = C^\tau\rho C\tau D = \tau^{-1}C\tau\rho C\tau D = \tau^{-1}CC\tau D = \tau^{-1}C\tau D = C^\tau D = DD = D. \tag{7}$$

Since $(ii) \Rightarrow (iii)$, we also have $C = D^\rho$ and $D\rho\tau = D$, whence we obtain analogously that

$$C\tau D\rho C = C. \tag{8}$$

From Equations (7) and (8) it follows that the maps $C\tau\restriction \mathrm{Cl}(D) : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ and $D\rho\restriction \mathrm{Cl}(C) : \mathrm{Cl}(C) \to \mathrm{Cl}(D)$ are inverses to each other. Since both are monotone, they are lattice isomorphisms.

Furthermore, we have that $C\tau D = C\tau C^\tau = C\tau$ and $D\rho C = D\rho D^\rho = D\rho$. Hence, τ and ρ induce a similarity between C and D, in virtue of Lemma 40. □

Definition 43. Let C and D be structural closure operators on M-sets $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$. An *equivalence* between D and C is an isomorphism $F : \mathbf{Cl}(D, M) \to \mathbf{Cl}(C, M)$. If such an isomorphism exists, C and D are said to be *equivalent*. Furthermore, F is said to be *induced* by translations τ and ρ if it is induced by τ and ρ as a similarity.

That is to say, an equivalence is a structural similarity or, in other words, a bijective structural representation. Then, Proposition 27 implies:

Theorem 44. *Let C and D be structural closure operators on M-sets $\langle A, \cdot\rangle$, $\langle B, \cdot\rangle$. If F is a similarity between D and C that is induced by translations τ and ρ, and one of them is structural, then F is an equivalence.* □

Finally, we characterize equivalences between graded M-sets having graded variables.

Theorem 45. *Let M be a monoid, $\langle A, \cdot, \jmath\rangle$ and $\langle B, \cdot, \imath\rangle$ two graded M-sets with graded variables, and C and D structural closure operators on $\langle A, \cdot\rangle$ and $\langle B, \cdot\rangle$, respectively. Then every equivalence between C and D is induced by structural translations τ and ρ. Moreover, if D is finitary, τ can be taken to be also finitary; and mutatis mutandis with C and ρ.*

Proof. If $F : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ is an equivalence between C and D, then in particular it is a structural join representation of D in C, and since $\langle B, \cdot, \imath\rangle$ has a graded variable, in virtue of Theorem 36, F is induced by a structural translation $\tau \in \mathrm{Trans}(B, A)$. Analogously, there is a structural translation $\rho \in \mathrm{Trans}(A, B)$ that induced F^{-1}. The finitarity part also follows from Theorem 36. □

9 Representations and extensions of structural closure operators

According to [BP89] a class of $\mathcal{L}$-algebras K is an *algebraic semantics* for a sentential logic $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$, if $\vdash_{\mathcal{S}}$ can be interpreted in $\vDash_{\mathsf{K}}$ in the following sense: there exists a finite set of equations Π in a single sentential variable x such that, for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \bigcup_{\psi \in \Gamma} \Pi[\psi/x] \vDash_{\mathsf{K}} \Pi[\varphi/x],$$

where $\Pi[\eta/x]$ stands for the set of all equations resulting from replacing x by the formula η in every equation of Π. The equations in Π are called the *defining equations*.

Following [Cze01] we extend this definition and do not require the set Π to be finite. Since sentential logics are based on M-sets with variables, every translation from $Fm_{\mathcal{L}}$ to $Eq_{\mathcal{L}}$ is determined by a set of equations $\Pi \subseteq Eq_{\mathcal{L}}$ (see Section 7 for the explanation in the case of Gentzen systems). This allows us to reformulate this definition saying that K is an algebraic semantics for $\mathcal{S}$ if, and only if, there exists a structural translation $\tau \in \mathrm{Trans}(Fm_{\mathcal{L}}, Eq_{\mathcal{L}})$ such that

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \tau\Gamma \vDash_{\mathsf{K}} \tau\{\varphi\}.$$

That is to say, τ induces a structural representation of $\vdash_{\mathcal{S}}$ in $\vDash_{\mathsf{K}}$. Moreover, since the $M(\mathcal{L})$-set of fomulas $\langle Fm_{\mathcal{L}}, \cdot\rangle$ has graded variables, every structural representation of $\vdash_{\mathcal{S}}$ in $\vDash_{\mathsf{K}}$ is induced by a structural translation. Therefore, K is an algebraic semantics for $\mathcal{S}$ if, and only if, there exists a structural representation of $\vdash_{\mathcal{S}}$ in $\models_{\mathsf{K}}$.

Thus, we can extend the notion of algebraic semantics to Gentzen systems, hypersequent systems, etc. A class of $\mathcal{L}$-algebras K is an *algebraic semantics* for a structural closure operator on an $M(\mathcal{L})$-set $\langle A, \cdot\rangle$ if, and only if, there exists a structural translation τ that induces a representation of $\langle A, \cdot\rangle$ in $\vDash_{\mathsf{K}}$.

It is well known that the set of congruences of the formula algebra $\mathrm{Co}(\mathbf{Fm}_{\mathcal{L}})$ is a structural closure system on $Eq_{\mathcal{L}}$, and that equational consequences of the form $\vDash_{\mathsf{K}}$ for some class of algebras K coincide with the extensions of the Birkoff consequence B_0, the closure operator associated with $\mathrm{Co}(\mathbf{Fm}_{\mathcal{L}})$. Thus, a sentential logic $\mathcal{S}$ has an algebraic semantics if, and only if, there exists a structural representation of $\vdash_{\mathcal{S}}$ in an extension of B_0.

In [BR03] it was proved that if a deductive system $\mathcal{S}$ has an algebraic semantics, then so does any extension of $\mathcal{S}$, with the same defining equations. We are looking for a generalization of this theorem in the following terms: if a structural closure operator D is representable by a structural translation, then every extension of D is representable by the same translation.

Recall from Section 4 the notion of τ-filter of a closure operator D on a set B, where $\tau \in \mathrm{Trans}(B, A)$, and the closure operator D_τ on A associated with the closure system of all τ-filters of D, τ-$\mathcal{F}il^D$. We will see that if D is representable in some closure operator on A by τ, then D is representable in D_τ, as well.

Proposition 46. *Let D be closure operators on B, and $\tau \in \mathrm{Trans}(B, A)$. If D is representable by τ, then D is representable in D_τ by τ.*

Proof. Suppose that D is representable by τ in a closure operator C. Then $D = C^\tau$, and in particular $D \leqslant C^\tau$. By applying Proposition 14, $D_\tau \leqslant C$. Thus,

$$D \leqslant (D_\tau)^\tau \leqslant C^\tau = D. \qquad \square$$

In what follows, if $\tau \in \mathrm{Trans}(B, A)$ and C and D are closure operators in A and B, respectively, then we denote the map $C\tau \restriction \mathrm{Cl}(D) : \mathrm{Cl}(D) \to \mathrm{Cl}(C)$ by $C\tau|_D$.

Theorem 47. *Let D be a closure operators on B, $\tau \in \mathrm{Trans}(B, A)$, and E an extension of D_τ. Then D is representable in E by τ if and only if $E\tau|_D$ is injective.*

Proof. Note that if D is representable in E by τ, then $D = E^\tau$, and thus $E\tau|_D = [E\tau]$, which is injective as we saw in Lemma 16.

Let suppose now that $D \neq E^\tau$. Since by hypothesis, $D_\tau \leqslant E$, therefore $D \leqslant (D_\tau)^\tau \leqslant E^\tau$, and hence $D \lneq E^\tau$, which implies that $\mathrm{Cl}(E^\tau) \subsetneq \mathrm{Cl}(D)$. Let $T \in \mathrm{Cl}(D) \setminus \mathrm{Cl}(E^\tau)$. Hence, T and $E^\tau T$ are two different D-closed sets, but $E\tau|_D T = E\tau T = E\tau E^\tau T = E\tau|_D E^\tau T$. Therefore, $E\tau|_D$ is not injective. $\square$

Theorem 48. *Let D and C be closure operators on B and A, respectively, $\tau \in \mathrm{Trans}(B, A)$. If D is representable in C by τ, then every extension of D is also representable in an extension of C by τ.*

Proof. Suppose that D' is an extension of D and let $E = (D')_\tau \vee C$. Since by hypothesis D is representable by τ in C, then $D = C^\tau$. We show now that, in this situation, $E\tau|_{D'} = (C\tau|_D)\restriction \mathrm{Cl}(D')$.

Let T be an arbitrary D'-closed set. We have that $\tau^{-1}C\tau = C^\tau T = DT = T \in \mathrm{Cl}(D')$, which implies that $C\tau T$ is a τ-filter of D'. Moreover, obviously $C\tau T \in \mathrm{Cl}(C)$. Hence, $C\tau T$ is an E-closed set, and then,

$$E\tau|_{D'}T = E\tau T = EC\tau T = C\tau T = (C\tau|_D)\restriction \mathrm{Cl}(D')T.$$

Since $D = C^\tau$, we have $C\tau|_D = [C\tau]$ which is injective, and then $E\tau|_{D'} = (C\tau|_D)\restriction \mathrm{Cl}(D')$ is also injective. Thus, since E is an extension of $(D')_\tau$, in virtue of Theorem 47, D' is representable in E by τ, and obviously E is an extension of C. ☐

Note that, in virtue of Proposition 27, a "structural" version of the preceding theorem can be proved:

Theorem 49. *Let D and C be structural closure operators on M-sets $\langle B, \cdot\rangle$ and $\langle A, \cdot\rangle$, respectively, and $\tau \in \mathrm{Trans}(B, A)$ a structural translation. If D is representable in C by τ, then every structural extension of D is also representable in a structural extension of C by τ.* ☐

As a consequence of this theorem, if a Gentzen system (hypersequent system, etc.) has an algebraic semantics K, then every of its extensions also has an algebraic semantics with the same translation. In particular we have:

Corollary 50 (Thm. 2.15 of [BR03])**.** *If a deductive system $\mathcal{S}$ has an algebraic semantics, then so does any extension of $\mathcal{S}$, with the same defining equations.*

Proof. Let $D = Cn_{\mathcal{S}}$ be the closure operator associated with $\vdash_{\mathcal{S}}$. If $\mathcal{S}$ has an algebraic semantics, then there exists a class of algebras K such that D is representable in $C = Cn_{\mathsf{K}}$, the closure operator associated with $\vDash_{\mathsf{K}}$. If $\mathcal{S}'$ is an extension of $\mathcal{S}$, then $D' = Cn_{\mathcal{S}'}$ is a structural extension of D, and in virtue of Theorem 49, it is representable in a structural extension of C, E, with the same defining equations. Since every element of $\mathrm{Cl}(C)$ is a congruence of the formula algebra, then so is every element in $\mathrm{Cl}(E)$, and thus, there exists a class of algebras K' such that E is the closure operator associated with $Cn_{\mathsf{K}'}$. Therefore, K' is an algebraic semantics of $\mathcal{S}'$, with the same defining equations. ☐

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