A Result about Atoms Revisited

Francisco M. García Olmedo
Antonio J. Rodríguez Salas
Departamento de Álgebra
Universidad de Granada
18071 Granada, España

Abstract

In this paper a new temporal algebra constructed on a product of finite ones is introduced. Using only algebraic tools, we show that every atom of the free temporal algebra $\mathbf{F}_t(X)$ must be less or equal than $g0 \wedge h0$. This result will be used to give a new algebraic proof of Theorem 1.1 in [1]: for each $n \in \omega$, $\mathbf{F}_t(n)$ has 2^n atoms.

Introduction

This paper is inspired by the excellent article [1] about atoms of tense algebras. F. Bellissima and others authors (see, for example, [6] and [7]) deal, in more or less generality, with the study of algebra of temporal logic; however tools and arguments in these works are essentially drawn from the logic and its semantics of frames although sometimes the goal is to obtain results on the algebra, as is the case in [1]. It can give the impression that Algebra does not have the resources to do analysis by himself; nevertheless this is not the case. Authors of this paper have suggested how powerful is the language of Algebra to study logics such as temporal logic (see, for example, [4] and [5]). In particular theorem 1.1 of [1] can be proved using purely algebraic techniques —on the language of [3]— without an appeal to the intuition about abbreviated schemes and without reasonings on complicated temporal structures; this is the aim of our work. Our proof of the result will be the third, since in [6] and [7] one can find implicitly a non-algebraic outline of a proof which is very different from ours and from that of F. Bellissima.

The proof of Theorem 1.1 in [1] exposed here is presented in two independent parts: the technical (Sections 2 and 3) and the conclusion (Section 4). The critical point in the technical reasoning is based on a certain product of finite temporal algebras in which the definition of g and h has been specially changed maintaining the structure of temporal algebra in the product. The main —and key— idea is to intersect in an appropriate manner with previously selected antiatoms.

The first section is a summarized compilation of preliminary basic concepts and the main theorem of [4]. Knowledge of both [4] and [5] is assumed in the reader of this paper. We have not found an explicit treatment of simple temporal algebras in the literature, so we give here a very useful characterization for them. In the second section we define the "modified product" of finite temporal algebras and we give its essential properties; this leads us to Theorem 2.7. For this we use the concept of degree of a formula given in Definition 2.2 and we analyze the essential differences with the notions of degree of [1] and [6]. Since we need Theorem 2.7 in full generality, in the third section we include the concept of simplicity in our argument; so we reach the second case, namely, Theorem 3.7. As a consequenze we obtain that every atom of the free temporal algebra $\mathbf{F}_t(X)$ must be less or equal than $g0 \wedge h0$ (Corollary 3.8). This result will be used in an essential manner in the fourth section, where we first show that every temporal algebra is the product of two of a specific kind. Then, as a straightforward corollary, we obtain that in case X is finite with n elements, $\mathbf{F}_t(X)$ has 2^n atoms. The "structural explanation" of this fact is clearly exposed in [1]; nevertheless the algebraic reasons can be found in the present paper.

We think our results, and especially our proofs, show some of the advantages of studying the algebraizable polymodal logics from the point of view, language, and techniques of Universal Algebra.

1 Preliminary

The paper deals with temporal algebras. A *temporal algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, g, h, 1\rangle$ of type $\langle 2, 2, 1, 1, 1, 0\rangle$ such that:

T.1) $\langle A, \wedge, \vee, \neg, 1\rangle$ is a Boolean algebra.

T.2) Both g and h are $\wedge$-morphisms (i.e. $k(a \wedge b) = ka \wedge kb$, for $k \in \{g, h\}$).

T.3) The equivalence [$ga \vee b = 1$ if, and only if, $a \vee hb = 1$] is satisfied.

T.4) $g1 = h1 = 1$.

As usual, we also consider in A the operators $p = \neg h \neg$ and $f = \neg g \neg$, as well as the unary operators L and M defined respectively by $Lx = hx \wedge x \wedge gx$ and $Mx = px \vee x \vee fx$. In the sequel we will write $g_*(X)$ for $\{gx \colon x \in X\}$ and similarly for h.

The class T of temporal algebras is a variety. In the sequel we will denote by $\mathbf{F}(X)$ (resp. $\mathbf{F}_t(X)$) the algebra of terms of type $\langle 2, 2, 1, 1, 1, 0\rangle$ over (resp. the free temporal algebra freely generated by) the set X. The universe of $\mathbf{F}(X)$ (resp. $\mathbf{F}_t(X)$) is denoted by $F(X)$ (resp. $F_t(X)$). $\mathbf{F}_t(X)$ is a quotient of $\mathbf{F}(X)$ by certain congruence $\theta_t(X)$, or simply θ. Hence, the elements of $F_t(X)$ are the quotient classes α/θ, with $\alpha \in F(X)$. If we represent by π_θ, or simply π, the epimorphic projection of $\mathbf{F}(X)$ onto $\mathbf{F}_t(X)$ and i is the inclusion map of X in $F(X)$, it is well known that for all temporal algebras $\mathbf{A}$ and for all mappings $v : X \longrightarrow A$ there are unique morphisms $\bar{v} : \mathbf{F}(X) \longrightarrow \mathbf{A}$ and $\tilde{v} : \mathbf{F}_t(X) \longrightarrow \mathbf{A}$ such that $v = \bar{v} \circ i$ and $\bar{v} = \tilde{v} \circ \pi$. If $\alpha \in F(X)$, we will write indistinctly $\pi(\alpha)$ or α/θ.

The characterization of simple temporal algebras follows immediately from the standard characterization of congruences by means of Boolean filters closed under the operators g and h; nevertheless we will sum up the most useful for our purpose. In any simple temporal algebra an element different from 1 can be diminished progressively to 0 by means of iterated applications of the operator L. Furthermore, this is only possible in a simple algebra. Of course in this statement L can be changed to M, whenever the roles of 1 and 0 are interchanged. In a temporal algebra $\mathbf{A}$ this is essentially due to two obvious facts:

1. If $\mathbf{A}$ is simple, $X \in \mathbf{P}(A) \setminus \{\emptyset, \{1\}\}$, and $a \in A$, then there are $Y_a \in \mathbf{P}_\omega(X)$ and $n_a \in \omega$ such that $L^{n_a}(\bigwedge Y_a) \leq a$, where $\mathbf{P}(A)$ (resp. $\mathbf{P}_\omega(X)$) is the set of all subsets of A (resp. of all finite subsets of X)
2. If $\mathbf{A}$ is a temporal algebra, then the statements:
   (a) For all $X \in \mathbf{P}(A) \setminus \{\emptyset, \{1\}\}$ and $a \in A$ there are $Y_a \in \mathbf{P}_\omega(X)$ and $n_a \in \omega$ such that $L^{n_a}(\bigwedge Y_a) \leq a$.
   (b) For all $x \in A \setminus \{1\}$ there is $n_x \in \omega$ such that $L^{n_x} x = 0$.

   are equivalent.

The announced theorem is the following.

Theorem 1.1. *Let* $\mathbf{A}$ *be a temporal algebra. Then the statements:*

1. $\mathbf{A}$ *is simple.*
2. *For every* $a \in A \setminus \{1\}$ *there is* $n_a \in \omega$ *such that* $L^{n_a} a = 0$.
3. *For every* $a \in A \setminus \{0\}$ *there is* $m_a \in \omega$ *such that* $M^{m_a} a = 1$.
4. *For all non-trivial temporal algebras* $\mathbf{B}$, *if* $\phi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a morphism of temporal algebras, then* ϕ *is a monomorphism.*

are equivalent.

The paper [4] is devoted to the "algebraic finite model property". We will use here the following result proved there: let $\alpha \in F(X)$ such that $\pi(\alpha) \neq 1$; then there exists a temporal valuation w_α over a finite temporal algebra $\mathbf{A}_\alpha$ such that $\tilde{w}_\alpha \neq 1$.

A filter F (resp. an ideal I) of $\langle A, \wedge, \vee, \neg, 1\rangle$ is a *temporal filter* (resp. *ideal*) iff $Lx \in F$ (resp. $Mx \in I$) whenever $x \in F$ (resp. $x \in I$). The symbol $Tsp(\mathbf{A})$ will represent the set of temporal filters of $\mathbf{A}$. If $a \in A$, we henceforth will represent the set $\{x \in A \colon a \leq x\}$ (resp. $\{x \in A \colon x \leq a\}$) by $[a, 1]$ (resp. $[0, a]$).

2 The Key Construction

In this section we will introduce a particular example of finite temporal algebra constructed from the product of two finite temporal algebras, and then we will we give some technical results about this construction. Actually the construction is done by modifying "smoothly" the temporal operations of the product algebra.

Our aim in this paper is to prove that if $\alpha \in F(X)$ is such that $\pi(\alpha)$ is an atom, then it is bounded by $g0 \wedge h0$. Implicitly in [2], F. Bellissima gives a proof of this fact. We offer here a direct, constructive and purely algebraic proof. In the following we will use the shorthand ω^* for $\omega \setminus \{0\}$.

Definition 2.1. Let $\mathbf{A}$ and $\mathbf{B}$ be finite temporal algebras. Let us assume that: $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $k \in \omega^*$, and $b \leq pa$. We define in $A^k \times B$ the unary operations g and h as follows. For all, $\langle y_0, \ldots, y_k\rangle \in A^k \times B$:

$$\pi_i(h\langle y_0, \ldots, y_k\rangle) = \begin{cases} hy_i, & \text{if } i = 0 \text{ or } (i > 0 \text{ and } a \leq y_{i-1}), \\ hy_i \wedge \neg b, & \text{if } 1 \leq i \leq k-1 \text{ and } a \not\leq y_{i-1}, \\ hy_i \wedge \neg c, & \text{if } i = k \text{ and } a \not\leq y_{k-1}. \end{cases}$$

$$\pi_i(g\langle y_0, \ldots, y_k\rangle) = \begin{cases} gy_i, & \text{if } i = k \text{ or } (i = k-1 \text{ and } c \leq y_k) \\ & \text{or } (i < k-1 \text{ and } b \leq y_{i+1}), \\ gy_i \wedge \neg a, & \text{if } (i < k-1 \text{ and } b \not\leq y_{i+1}) \\ & \text{or } (i = k-1 \text{ and } c \not\leq y_k). \end{cases}$$

Next we give our temporal algebra, in fact the main tool in this paper. Let $\mathbf{A}$ and $\mathbf{B}$ be two finite temporal algebras. Suppose that $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $k \in \omega^*$, and $b \leq pa$. The algebra $\mathbf{A}^k_{a,b} \times \mathbf{B}_c = \langle A^k \times B, \wedge, \vee, \neg, g, h, 1\rangle$; where $\wedge$, $\vee$, $\neg$, and 1 are the operations component-wise on the product and the operations g y h are as in Definition 2.1, is a temporal algebra. In fact, for this algebra the operations f and p are as follows:

$$\pi_i(p\langle y_0, \ldots, y_k\rangle) = \begin{cases} py_i, & \text{if } i = 0 \text{ or } (1 \leq i \leq k-1 \text{ and } a \not\leq y_{i-1}) \\ & \text{or } (i = k \text{ and } a \not\leq y_{k-1}), \\ py_i \vee b, & \text{if } 1 \leq i \leq k-1 \text{ and } a \leq y_{i-1}, \\ py_i \vee c, & \text{if } i = k \text{ and } a \leq y_{k-1} \end{cases}$$

$$\pi_i(f\langle y_0, \ldots, y_k\rangle) = \begin{cases} fy_i, & \text{if } (0 \leq i < k-1 \text{ and } b \not\leq y_{i+1}) \\ & \text{or } (i = k-1 \text{ and } c \not\leq y_k) \text{ or } i = k, \\ fy_i \vee a, & \text{if } (0 \leq i < k-1 \text{ and } b \leq y_{i+1}) \\ & \text{or } (i = k-1 \text{ and } c \leq y_k). \end{cases}$$

Remark 2.1. To shorten, we will assume in this section that $\mathbf{A}$ and $\mathbf{B}$ are both finite temporal algebras. Moreover, when we write $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ we presupose that: $a, b \in Atm(\mathbf{A})$, $c \in Atm(\mathbf{B})$, $b \leq pa$, $k \in \omega^*$, and, finally, that the temporal operations are according to Definition 2.1. Sometimes we will represent the universe of $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ by $A^k_{a,b} \times B_c$, though this universe is in fact the set $A^k \times B$.

Lemma 2.1. *In $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ the equality $g0 = \langle g0, \overset{k+1)}{\ldots}, g0\rangle$ holds.*

Proof. We have the equality $g0 = \langle \neg a \wedge g0, \overset{k)}{\ldots}, \neg a \wedge g0, g0\rangle$. Since $b \leq pa$, we have that $a \leq \neg g0$, or equivalently, $g0 \leq \neg a$; hence $\neg a \wedge g0 = g0$. □

Lemma 2.2. *Let $k \in \omega$ such that $2 \leq k$ and $\langle y_0, \ldots, y_k\rangle \in A^k_{a,b} \times B_c$. If there is $m \in \omega^*$ such that $m \leq k-1$ and $y_i = y_0$, for all $0 \leq i \leq m$, then for all $0 \leq i \leq m$, $\pi_i(h\langle y_0, \ldots, y_k\rangle) = \pi_0(h\langle y_0, \ldots, y_k\rangle)$ and for all $0 \leq i \leq m-1$, $\pi_i(g\langle y_0, \ldots, y_k\rangle) = \pi_0(g\langle y_0, \ldots, y_k\rangle)$*

Proof. From the hypotheses of the lemma we have $m \leq k-1$. Suppose that $j \leq k-1$ and that $a \nleq y_j$ or, equivalently, that $y_j \leq \neg a$. Since $b \leq pa$, it follows that $hy_j \leq \neg b$, that is, $hy_j \wedge \neg b = hy_j$. Hence, if $0 \leq i \leq m$, then $\pi_i(h\langle y_0, \ldots, y_k\rangle) = hy_0$, from which the first statement follows. Moreover, according to the definition of g in $\mathbf{A}^k_{a,b} \times \mathbf{B}_c$ we have, for all $0 \leq i \leq m$,

$$\pi_i(g\langle y_0, \ldots, y_k\rangle) = \begin{cases} gy_0, & \text{if } (i = k-1 \text{ and } c \leq y_k) \\ & \text{or } (i < k-1 \text{ and } b \leq y_{i+1}), \\ gy_0 \wedge \neg a, & \text{if } (i = k-1 \text{ and } c \nleq y_k) \\ & \text{or } (i < k-1 \text{ and } b \nleq y_{i+1}). \end{cases}$$

From this it follows that each $0 \leq i \leq m-1$ satisfies $\pi_i(g\langle y_0, \ldots, y_k\rangle) = gy_0$ or $\pi_i(g\langle y_0, \ldots, y_k\rangle) = gy_0 \wedge \neg a$, depending on whether $b \leq y_0$ holds or not. □

We now need a notion of the degree of a formula. This notion is, in fact, the number of g-symbols in a formula.

Remark 2.2. In the following we will assume that X is the finite set $\{x_1, \ldots, x_n\}$, where $n \in \omega^*$.

Definition 2.2. Let $\alpha \in F(X)$, the *degree* of α, $\deg(\alpha)$, is defined as follows:

$$\deg(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ \deg(\beta), & \text{if } \alpha = \neg\beta, \\ \max\{\deg(\beta), \deg(\gamma)\}, & \text{if } \alpha = \beta \wedge \gamma, \\ \max\{\deg(\beta), \deg(\gamma)\}, & \text{if } \alpha = \beta \vee \gamma, \\ \deg(\beta), & \text{if } \alpha = h\beta, \\ \deg(\beta) + 1, & \text{if } \alpha = g\beta. \end{cases}$$

Note that the concept of "degree" in [6] is defined on g and h, whereas it seems that in [1], F. Bellisima defines "degree" counting "inmediate alternation" between f and p, occurrences of 0 and 1, and occurrences of elements in X. Our notion of degree is essentially different, based only on g, in order to make induction easier; actually we don't need to count any other feature in α. Consequently the arguments and proofs in this paper belong to a different circle of ideas than that of [6] and [1].

The proof of the following theorem follows by induction over the complexity of the formula α. It is straightforward after the definition of degree and Lemma 2.2.

Theorem 2.3. *Let $v : X \longrightarrow \mathbf{A}$ be a temporal valuation and $\bar{v}$ its extension as a morphism to $\mathbf{F}(X)$. Let us consider the temporal valuation $w\colon X \longrightarrow A^k_{a,b} \times B_c$ defined, for all $1 \leq i \leq n$, by $w(x_i) = \langle v(x_i), v(x_i), \ldots, v(x_i), 0\rangle$. For all $\alpha \in F(X)$ and $0 \leq j \leq k - (\deg(\alpha) + 1)$, $\pi_j(\bar{w}(\alpha)) = \bar{v}(\alpha)$ (or equivalently, $\pi_j(\tilde{w}(\pi(\alpha))) = \tilde{v}(\pi(\alpha))$), whenever $\deg(\alpha) < k$.*

We will give some results about the formulas α such that $p\pi(\alpha) \neq 0$. The first of these results is a particular case of the "algebraic finite model property".

Theorem 2.4. *Let $\alpha \in F(X)$ such that $p\pi(\alpha) \neq 0$. There exists a finite simple temporal algebra $\mathbf{A}_\alpha$ and a morphism $\tilde{v}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ such that $\tilde{v}(p\pi(\alpha)) \neq 0$ and $\tilde{v}(\pi(\alpha)) \neq 0$.*

Proof. Since $\neg p\pi(\alpha) \neq 1$, there are (see [4]) a finite temporal algebra $\mathbf{A}$ and a morphism $\tilde{u}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}$ such that $\tilde{u}(\neg p\pi(\alpha)) \neq 1$, so $\tilde{u}(p\pi(\alpha)) \neq 0$. Since $\mathbf{A}$ is finite, then it is isomorphic to a product of finite simple temporal algebras (see [5]). Composing $\tilde{u}$ with the canonical projection over the convenient simple factor of $\mathbf{A}$, $\mathbf{A}_\alpha$, we obtain the mapping $\tilde{v}$ of the statement. It is clear that $\tilde{v}(p\pi(\alpha)) \neq 0$. Therefore the equality $\tilde{v}(\pi(\alpha)) = 0$ is imposible. □

In the following we will consider that $\pi(\alpha) \in F_t(X)$ satisfies $p\pi(\alpha) \neq 0$. So $\mathbf{A}_\alpha$ and $\tilde{v}$ will represent the algebra and the morphism of Theorem 2.4. We will assume too that $a, b \in Atm(\mathbf{A}_\alpha)$ are such that $a \leq \tilde{v}(\pi(\alpha))$ and $b \leq \tilde{v}(p\pi(\alpha))$. Moreover, if $\mathbf{B}$ is a temporal algebra, we define the map $\tau\colon B \longrightarrow B$ by the equality $\tau(y) = fpy$. In the following we will use the symbol $\mathbf{E}_0$ to represent the simple temporal algebra $\langle B, \wedge, \vee, \neg, k, k, 1\rangle$, where $B = \{0, 1\}$ and $k\colon B \to B$ is the map defined by $k(0) = k(1) = 1$.

Theorem 2.5. *Let* $\mathbf{A}$ *be a finite temporal algebra. In the algebra* $\mathbf{A}_{a,b}^{m+1} \times (\mathbf{E}_0)_c$*, where* $m \in \omega^*$ *and* $c = 1$*, the equality:*

$$\tau^i(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) = \langle 0, \overset{m-i)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^i(a), 0\rangle \tag{1}$$

holds for all $0 \leq i \leq m$.

Proof. The proof is by induction on i. For $i = 0$ the statement is obviously true. Let us assume that the theorem holds for j and let us take $i = j + 1$. It is straightforward to show that $a \leq \tau a$. Since τ is an increasing funtion, $\{\tau^j(a)\}_{j\in\omega}$ increases with j, hence for all $j \in \omega$, $a \leq \tau^j a$ and so $b \leq p\tau^j(a)$. By definition of f and p we have:

$$\begin{aligned}
\tau^i(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) &= fp\langle 0, \overset{m-j)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^j(a), 0\rangle \\
&= f\langle 0, \overset{m-j)}{\ldots}, 0, pa, p\tau(a), p\tau^2(a), \ldots, p\tau^j(a), 1\rangle \\
&= \langle 0, \overset{m-j-1)}{\ldots}, 0, a, a \vee \tau(a), a \vee \tau^2(a), \ldots, a \vee \tau^{j+1}(a), 0\rangle \\
&= \langle 0, \overset{m-i)}{\ldots}, 0, a, \tau(a), \tau^2(a), \ldots, \tau^i(a), 0\rangle
\end{aligned}$$

which is just what we wanted to show. □

Equation (1) is valid when $i = m$, even if $m = 0$; therefore the following corollary holds.

Corollary 2.6. *Let* $\mathbf{A}$ *be a finite temporal algebra. In the algebra* $\mathbf{A}_{a,b}^{m+1} \times (\mathbf{E}_0)_c$*, where* $m \in \omega$ *and* $c = 1$*, the equality* $\tau^m(\langle 0, \overset{m)}{\ldots}, 0, a, 0\rangle) = \langle a, \tau(a), \tau^2(a), \ldots, \tau^m(a), 0\rangle$ *holds.*

Theorem 2.7. *Let* $\alpha \in F(X)$ *be such that* $p\pi(\alpha) \neq 0$*,* $\mathbf{A}_\alpha$ *a finite simple temporal algebra,* $\tilde{v} \colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ *such that* $\tilde{v}\big(p\pi(\alpha)\big) \neq 0$*, and consider the element* $\tau^m(fg0) \in F_t(X)$*, where* $m = \deg(\alpha)$*. If the equality* $g0 = 0$ *holds in* $\mathbf{A}_\alpha$*, then* $0 < \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$.

Proof. We have $\tilde{v}\big(\tau^m(fg0)\big) = \tau^m\big(f\tilde{v}(g0)\big) = \tau^m(f0) = 0$. Since $\tilde{v}\big(\pi(\alpha)\big) \neq 0$, the inequality $\pi(\alpha) \leq \tau^m(fg0)$ is not possible; therefore $0 \leq \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$. All we need to do is to show that $0 < \pi(\alpha) \wedge \tau^m(fg0)$. For this, take the algebra $\mathbf{A}_{a,b}^k \times (\mathbf{E}_0)_c$, where $c = 1$ and $k = \deg(\alpha) + 1$, and define the temporal valuation $w \colon \{x_1, \ldots, x_n\} \longrightarrow A_{a,b}^k \times B_c$ by the equality $w(x_i) = \langle v(x_i), \overset{k)}{\ldots}, v(x_i), 0\rangle$. According to Theorem 2.3, for all $0 \leq j \leq k - \big(\deg(\alpha) + 1\big)$, $\pi_j\big(\tilde{w}(\pi(\alpha)\big) = \tilde{v}\big(\pi(\alpha)\big)$ holds. In particular $\pi_0\big(\tilde{w}\big(\pi(\alpha)\big)\big) = \tilde{v}\big(\pi(\alpha)\big)$, hence $\langle a, 0, \overset{k)}{\ldots}, 0\rangle \leq \tilde{w}\big(\pi(\alpha)\big)$. Since $g0 = 0$ holds in $\mathbf{A}_\alpha$, the value of $fg0$ in $\mathbf{A}_{a,b}^k \times (\mathbf{E}_0)_c$ is $\langle 0, \overset{k-1)}{\ldots}, 0, a, 0\rangle$. According to Corollary 2.6 we have $\langle a, 0, \overset{k)}{\ldots}, 0\rangle \leq \tilde{w}\big(\tau^m(fg0)\big)$. Since $\tilde{w}\big(\pi(\alpha)\big) \wedge \tilde{w}\big(\tau^m(fg0)\big) = \tilde{w}\big(\pi(\alpha) \wedge \tau^m(fg0)\big)$, it follows that $\pi(\alpha) \wedge \tau^m(fg0) \neq 0$, and so $0 < \pi(\alpha) \wedge \tau^m(fg0) < \pi(\alpha)$. □

3 Simplicity

In this section we introduce the hypothesis of simplicity in the general construction. The simplicity of $\mathbf{A}$ and $\mathbf{B}$ implies that $\mathbf{A}_{a,b}^k \times \mathbf{B}_c$ is simple and conversely. Nevertheless, this is more than what we need, so we will prove a weaker result. The following lemma indicates what can we expect about conmutativity between M and π_i. Its proof follows easily by induction on m using that $py_i \leq \pi_i(p\langle y_0, \ldots, y_k\rangle)$, $fy_i \leq \pi_i(f\langle y_0, \ldots, y_k\rangle)$ and that M is an increasing function.

Lemma 3.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite temporal algebras. For all* $m \in \omega$ *and* $0 \leq i \leq k$*, in* $\mathbf{A}_{a,b}^k \times \mathbf{B}_c$ *the inequality* $M^m\pi_i(\langle y_0, \ldots, y_k\rangle) \leq \pi_i(M^m\langle y_0, \ldots, y_k\rangle)$ *holds.*

Theorem 3.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two finite simple temporal algebras and* $y \in A \setminus \{0\}$*. If* $\langle y, \overset{k-1)}{\ldots}, y, z\rangle \in A_{a,b}^k \times B_c$ *then there exists* $m \in \omega^*$ *such that* $M^m\langle y, \overset{k-1)}{\ldots}, y, z\rangle = \langle 1, \overset{k-1)}{\ldots}, 1, z'\rangle$*, for some* $z' \in B$ *satisfying* $M^m z \leq z'$.

Proof. Since $\mathbf{A}$ is simple and $y \neq 0$, by Theorem 1.1, there exists $m \in \omega^*$ such that $M^m y = 1$. This and Lemma 3.1 imply that there is $z' \in B$ such that $M^m\langle y, \overset{k-1)}{\ldots}, y, z\rangle = \langle 1, \overset{k-1)}{\ldots}, 1, z'\rangle$. It is clear from Lemma 3.1 that $M^m z \leq z'$. □

Remark 3.1. If $\mathbf{B} = \mathbf{A}$ and $c = b$, the symbol $\mathbf{A}^{k+1}_{a,b}$ (resp. $\mathbf{A}$) will represent the algebra $\mathbf{A}^k_{a,b} \times \mathbf{A}_b$ (resp. $\mathbf{A}^1_{a,b}$).

Corollary 3.3. *Let a $\mathbf{A}$ be a finite simple temporal algebra and $y \in A \setminus \{0\}$. If $\langle y, \ldots, y\rangle \in A^k_{a,b}$ then there exists $m \in \omega^*$ such that $M^m\langle y, \ldots, y\rangle = \langle 1, \ldots, 1\rangle$ and $M^m y = 1$.*

Definition 3.1. Let $\mathbf{A}$ be a finite simple temporal algebra for which the condition $g0 \neq 0$ holds and there are $a, b \in Atm(\mathbf{A})$ such that $b \leq pa$. From Lemma 2.1 and Corollary 3.3, the set of all $j \in \omega$ such that $M^j g0 = 1$ at the same time in $\mathbf{A}^k_{a,b}$ and $\mathbf{A}$ is non-empty; so it is possible to take the minimum s of this set. Let

$$r = \min\{j \in \omega \colon \langle 0, \overset{k-1)}{\ldots}, 0, a\rangle \leq M^j g0\} \tag{2}$$

and

$$l = \max\{s - r, 1\}. \tag{3}$$

We define the value $t(\mathbf{A}, a)$ by the equality

$$t(\mathbf{A}, a) = \min\{2j \colon j \in \omega \text{ and } l + 1 \leq 2j\}. \tag{4}$$

Finally, let us define $\sigma \colon B \longrightarrow B$ by $\sigma(y) = ghy$.

Definition 3.2. Given $q \in \omega^*$, if B represents the set $\{0, 1\}$ we define the funtions $g, h \colon B^{2q} \longrightarrow B^{2q}$ as follows:

$$\pi_i(h\langle y_0, \ldots, y_{2q-1}\rangle) = \begin{cases} y_0 \wedge y_1, & \text{if } i = 0,\\ 0, & \text{if } i \text{ is odd},\\ y_{i-1} \wedge y_i \wedge y_{i+1}, & \text{otherwise.}\end{cases}$$

$$\pi_i(g\langle y_0, \ldots, y_{2q-1}\rangle) = \begin{cases} y_i, & \text{if } i \text{ is even},\\ y_{2q-2}, & \text{if } i = 2q-1,\\ y_{i-1} \wedge y_{i+1}, & \text{otherwise.}\end{cases}$$

Remark 3.2. Let $q \in \omega^*$. It is clear that the algebra $\mathbf{B}^{2q} = \langle B^{2q}, \wedge, \vee, \neg, g, h, 1\rangle$ is a temporal algebra. For $\mathbf{B}^{2q}$ the operations f and p are as follows:

$$\pi_i(f\langle y_0, \ldots, y_{2q-1}\rangle) = \begin{cases} y_i, & \text{if } i \text{ is even},\\ y_{2q-2}, & \text{if } i = 2q-1,\\ y_{i-1} \vee y_{i+1}, & \text{otherwise.}\end{cases}$$

$$\pi_i(p\langle y_0, \ldots, y_{2q-1}\rangle) = \begin{cases} y_0 \vee y_1, & \text{if } i = 0,\\ 0, & \text{if } i \text{ is odd},\\ y_{i-1} \vee y_i \vee y_{i+1}, & \text{otherwise.}\end{cases}$$

Moreover, in $\mathbf{B}^{2q}$ the equality $g0 = 0$ holds.

Lemma 3.4. *Let $\mathbf{A}$ be a finite simple temporal algebra such that $g0 \neq 0$ and let $a, b \in Atm(\mathbf{A})$ be such that $b \leq pa$. Let s be the least $j \in \omega$ satifying $M^j g0 = 1$ at the same time in $\mathbf{A}^k_{a,b}$ and $\mathbf{A}$. If r is the value given by (2), $c = \langle 1, 0, \ldots, 0\rangle$, and $q \in \omega^*$ then the following properties hold:*

1. *For all $j \leq r$ and $0 \leq i \leq k-1$, the value of $\pi_i\big(M^j g0\big)$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$ coincides with its value in $\mathbf{A}^k_{a,b}$.*
2. *For all $j \leq r$, $\pi_k\big(M^j g0\big) = 0$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$.*
3. *$\pi_{k-1}\big(M^r g0\big) = a$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$.*
4. *If $r < s$ and $s - r + 1 \leq 2q$, then for all $r < j \leq s$, $\pi_k\big(M^j g0\big) = \langle 1, \overset{j-r)}{\ldots}, 1, 0, \ldots, 0\rangle$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$.*
5. *If $2q$ is the number $t(\mathbf{A}, a)$, defined in (4), and $\langle y_0, \ldots, y_{2q-1}\rangle$ is $\pi_k\big(M^s g0\big)$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$, then $y_{2q-1} = 0$.*

Proof. We will prove the first two statements at the same time by induction. Actually, in the two algebras the values of $g0$ are $\langle g0, \overset{k)}{\ldots}, g0, 0\rangle$ and $\langle g0, \overset{k)}{\ldots}, g0\rangle$ respectively. So the properties follow in the case $j = 0$. Let assume that the properties hold for $j < r$. It is easy to verify the first one in the cases $0 \leq i < k-1$. When $i = k-1$, as $\pi_k(M^j g0) = 0$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$, the value of $\pi_{k-1}(fM^j g0)$ is $f\pi_{k-1}(M^j g0)$. In the case of $\mathbf{A}^k_{a,b}$, the value of $\pi_{k-1}(fM^j g0)$ is $f\pi_{k-1}(M^j g0)$; but, by the inductive hypothesis, $\pi_{k-1}(M^j g0)$ has the same value in the two algebras under cosideration, so the result holds for $k-1$. Since r is the least natural i such that $\langle 0, \overset{k-1)}{\ldots}, 0, a\rangle \leq M^i g0$, it follows that $a \not\leq \pi_{k-1}(M^j g0)$ in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$, and so $\pi_k(pM^j g0) = p0$ and $\pi_k(fM^j g0) = f0$. This implies that $\pi_k(M^{j+1} g0) = 0$. The third property is obvious from the first one. The fourth statement follows by induction too. For $j = r+1$ the result holds. Actually, according properties *2* and *3* we have $\pi_k\big(M^j g0\big) = 0$ and $\pi_{k-1}\big(M^r g0\big) = a$; then $\pi_k\big(pM^r g0\big) = \langle 1, 0, \ldots, 0\rangle$ and, furthermore, $\pi_k\big(M^{r+1} g0\big) = \langle 1, 0, \ldots, 0\rangle$. Let assume that $1 \leq i$, $r+i+1 \leq s$, and that the result holds for $r+i$. If $\langle y_0, \ldots, y_{2q-1}\rangle$ represents to $\pi_k\big(M^{r+i} g0\big)$, the inductive hypothesis means that $y_0 = \cdots = y_{r+i-1} = 1$ and $y_{r+i} = \cdots = y_{2q-1} = 0$. It is clear that

$$\pi_k\big(M^{r+i+1} g0\big) = p\langle y_0, \ldots, y_{2q-1}\rangle \vee \langle y_0, \ldots, y_{2q-1}\rangle \vee f\langle y_0, \ldots, y_{2q-1}\rangle$$

Hence all we need to do is to examine the right-hand member of this equality. Represent by $\langle z_0, \ldots, z_{2q-1}\rangle$ (resp. $\langle u_0, \ldots, u_{2q-1}\rangle$) the value $p\langle y_0, \ldots, y_{2q-1}\rangle$ (resp. $f\langle y_0, \ldots, y_{2q-1}\rangle$). As $M^{r+1} g0 \leq M^{r+i} g0$, then $y_0 = 1$, and so $z_0 = u_0 = 1$. In the other hand, $y_{2q-2} = 0$, hence $z_{2q-1} = u_{2q-1} = 0$. In case that $j \notin \{0, 2q-1\}$, the values of z_j and u_j are as follows:

case a) $y_j = 1$; in this case the result is obvious,

case b) $y_{j-1} = 1$ and $y_j = 0$; if j is even (resp. odd) then $z_j = 1$ (resp. $u_j = 1$),

case c) $y_{j-1} = 0$, $y_j = 0$; therefore $y_{j+1} = 0$ and so, if j is either even or odd, $z_j = 0$ and $u_j = 0$.

So the fourth property is established. The fifth follows from the fourth and the given definitions since the equality

$$\langle y_0, \ldots, y_{2q-1}\rangle = \begin{cases} \langle 0, \ldots, 0\rangle, & \text{if } s = r, \\ \langle 1, \ldots, 1, 0\rangle, & \text{if } s \neq r \text{ and } l \text{ is even}, \\ \langle 1, \ldots, 1, 0, 0\rangle, & \text{if } s \neq r \text{ and } l \text{ is odd}, \end{cases}$$

holds. □

Remark 3.3. We adopt in the sequel the following notational use. On one hand, for all $0 \leq i \leq 2q-1$ let z_i be the element of $\mathbf{B}^{2q}$ satisfying for all $0 \leq j \leq 2q-1$ the condition

$$\pi_j(z_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, if $y = \langle y_0, \ldots, y_k\rangle$ is in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$, then for every $0 \leq i \leq 2q-1$, $\vartheta_i(y)$ will be the abbreviation of $\pi_i(y_k)$.

Lemma 3.5. *Let* $\mathbf{A}$ *be a finite simple temporal algebra such that* $g0 \neq 0$ *and* $a, b \in Atm(\mathbf{A})$ *such that* $b \leq pa$. *Let* $q \in \omega^*$ *and let* c *be the atom* z_0 *of* $\mathbf{B}^{2q}$. *If* $\langle 0, \ldots, 0, z_{2q-1}\rangle \not\leq M^s g0$ *in* $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$ *for some* $s \in \omega$, *then* $\langle 0, \ldots, a, 0\rangle \not\leq \sigma^q\big(M^s g0\big)$.

Proof. Let us assume that $\langle 0, \ldots, 0, z_{2q-1}\rangle \not\leq M^s g0$ and use induction to show that for all $0 \leq i \leq q-1$, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \not\leq \sigma^i(M^s g0)$. In the case $i = 0$, the result follows directly from the hypotheses. Suppose that $0 \leq i < q-1$, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \not\leq \sigma^i(M^s g0)$, and nevertheless $\langle 0, \ldots, 0, z_{2(q-(i+1))-1}\rangle \leq \sigma^{i+1}(M^s g0)$. Hence $\langle 0, \ldots, 0, z_{2(q-i)-3}\rangle \leq gh\sigma^i(M^s g0)$. This implies that $\vartheta_{2(q-i)-3}\big(gh\sigma^i(M^s g0)\big) = 1$. According to the definition of g, since $2(q-i)-3$ is odd and different from $2q-1$, we conclude that $\vartheta_{2(q-i)-2}\big(h\sigma^i(M^s g0)\big) = 1$. Since $2(q-i)-2$ is even, $i < q-1$, therefore, by the definition of h, the equality $\vartheta_{2(q-i)-1}\big(\sigma^i(M^s g0)\big) = 1$ holds, or equivalently, $\langle 0, \ldots, 0, z_{2(q-i)-1}\rangle \leq \sigma^i(M^s g0)$ which is contradictory with the inductive hypothesis. In particular we have $\langle 0, \ldots, 0, z_1\rangle \not\leq \sigma^{q-1}(M^s g0)$. Let us suppose now that $\langle 0, \ldots, a, 0\rangle \leq gh\sigma^{q-1}\big(M^s g0\big)$, that is, $a \leq \pi_{k-1}\big(gh\sigma^{q-1}(M^s g0)\big)$. Since $a \neq 0$ and $z_0 = c$, we have that $z_0 \leq \pi_k\big(h\sigma^{q-1}(M^s g0)\big)$; hence $z_0 \leq h\pi_k\big(\sigma^{q-1}(M^s g0)\big)$ and, consequently, $z_1 \leq \pi_k\big(\sigma^{q-1}(M^s g0)\big)$, which is contradictory. Therefore, $\langle 0, \ldots, a, 0\rangle \not\leq \sigma^q\big(M^s g0\big)$. □

Theorem 3.6. *Let $\alpha \in F(X)$ be such that $p\pi(\alpha) \neq 0$ and both $\mathbf{A}_\alpha$ and $\tilde{v}$ the algebra and the morphism whose existence ensures Theorem 2.4. Let us assume that the condition $g0 \neq 0$ holds in $\mathbf{A}_\alpha$ and take the atoms $c = \langle 1, 0, \dots, 0\rangle$ and $d = \langle 0, \dots, 0, 1\rangle$ of $\mathbf{B}^{t(\mathbf{A}_\alpha, a)}$. If $\deg(\alpha) = m$, $k = m+1$, and $\tilde{w}$ is the extension to $\mathbf{F}_t(X)$ of the temporal valuation:*

$$w\colon X \longrightarrow \mathbf{A}^k_{a,b} \times \mathbf{B}_c^{t(\mathbf{A}_\alpha, a)}$$

defined by $w(x_i) = \langle v(x_i), \overset{k)}{\dots}, v(x_i), 0\rangle$, then the properties:

1. $\tilde{v}\big(\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big) = 1.$
2. $\langle 0, \overset{k)}{\dots}, 0, d\rangle \not\leq \tilde{w}(M^s g0).$
3. $\langle a, 0, \overset{k)}{\dots}, 0\rangle \not\leq \tilde{w}\big(\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big).$

hold.

Proof. According to the choice of s, the equality $M^s g0 = 1$ holds in $\mathbf{A}$. Since for all $j \in \omega$ we have that $\sigma^j(1) = 1$, it follows that $\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0) = 1$ and so $\tilde{v}\big(\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big) = 1$. The second property is an immediate consequence of part 5 in 3.4. For the third property we will show that, under the hypotheses of the theorem, if q represents to $t(\mathbf{A}_\alpha, a)/2 (\geq 1)$ and, for all $0 \leq i \leq k-1$, u_i is the element of $A^k \times B^{2q-1}$ satisfying for $0 \leq j \leq k$:

$$\pi_j(u_i) = \begin{cases} a, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

then in $\mathbf{A}^k_{a,b} \times \mathbf{B}^{2q}_c$ the condition $u_{m-i} \not\leq \sigma^{i+q}(M^s g0)$ holds for all $0 \leq i \leq m$. If $i = 0$ the result is true by Lemma 3.5. Let us suppose that $0 \leq i < m$ and that the result holds for i, that is, $u_{m-i} \not\leq \sigma^{i+q}(M^s g0)$. If $u_{m-i-1} \leq gh\sigma^{i+q}(M^s g0)$ then $a \leq \pi_{m-i-1}\big(gh\sigma^{i+q}(M^s g0)\big)$ and, so long as $m-i-1 < k-1$, we have

$$b \leq \pi_{m-i}\big(h\sigma^{i+q}(M^s g0)\big) \tag{5}$$

Since $m - i \leq k-1$, we deduce from (5) that $b \leq h\pi_{m-i}\big(\sigma^{i+q}(M^s g0)\big)$; but $b \leq pa$, hence $u_{m-i} \leq \sigma^{i+q}(M^s g0)$, which contradicts the inductive hypothesis. In particular, we have $u_0 \not\leq \sigma^{m+q}(M^s g0)$, that is to say $\langle a, 0, \overset{k)}{\dots}, 0\rangle \not\leq \sigma^{i+q}(M^s g0)$, which proves part 3. □

Theorem 3.7 is the analogous result to Theorem 2.7 for the case $g0 \neq 0$.

Theorem 3.7. *Let $\alpha \in F(X)$ such that $p\pi(\alpha) \neq 0$ and let us assume that in $\mathbf{A}_\alpha$ the condition $g0 \neq 0$ holds. Let $m = \deg(\alpha)$ and both s and $t(\mathbf{A}_\alpha, a)$ the values defined in 3.1. Then*

$$0 < \pi(\alpha) \wedge \sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0) < \pi(\alpha)$$

Proof. The morphism $\bar{v}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}_\alpha$ satisfies that $\bar{v}\big(\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big) = 1$ and that $\bar{v}\big(\pi(\alpha)\big) \neq 0$. So $0 < \bar{v}\big(\pi(\alpha) \wedge \sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big)$. For $\bar{w}\colon \mathbf{F}_t(X) \longrightarrow \mathbf{A}^k_{a,b} \times \mathbf{B}^{t(\mathbf{A}_\alpha,a)}$, where $\mathbf{A}_\alpha$ is written just as $\mathbf{A}$ and a is the selected atom satisfying $a \leq \bar{v}\big(\pi(\alpha)\big)$, the condition $\langle a, 0, \overset{k)}{\dots}, 0\rangle \not\leq \bar{w}\big(\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)\big)$ holds (see 3.6). Nevertheles, since $k = m+1$, Theorem 2.3 ensures that $\pi_0\big(\bar{w}(\pi(\alpha))\big) = \bar{v}\big(pi(\alpha)\big)$, therefore $\langle a, 0, \overset{k)}{\dots}, 0\rangle \leq \bar{w}\big(\pi(\alpha)\big)$. It follows that

$$\pi(\alpha) \not\leq \sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)$$

and so $0 < \pi(\alpha) \wedge \sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0) < \pi(\alpha)$. □

Corollary 3.8. *Let $\alpha \in F(X)$ be such that $\pi(\alpha) \in Atm\big(F_t(X)\big)$. Then $\pi(\alpha) \leq g0 \wedge h0$.*

Proof. If $\pi(\alpha) \not\leq g0 \wedge h0$, it is easy to show that $p\pi(\alpha) \neq 0$ or $f\pi(\alpha) \neq 0$ (in fact this logical disjunction is equivalent to the former condition). If $p\pi(\alpha) \neq 0$ then take β equal to $\tau^m fg0$ or $\sigma^{m+\frac{t(\mathbf{A}_\alpha,a)}{2}}(M^s g0)$ as needed (see Theorem 2.7 and Theorem 3.7). It follows that

$$0 < \pi(\alpha) \wedge \beta < \pi(\alpha),$$

hence the result. In case that $f\pi(\alpha) \neq 0$, it is feasable to give a dual reasoning. □

4 Atoms of Free Temporal Algebras

Speaking in the language of temporal structures, it is clear that every generalized structure is the disjoint union of two: $\langle W, R, P\rangle$ and $\langle V, S, Q\rangle$, where $R = \emptyset$ and S satisfying "for all $x \in V$ there exists $y \in V$ such that xSy or ySx". The algebraic meaning of this is that every temporal algebra is the product of two: the first one satisfying $g0 = 1$ and the second one satisfying $g0 \wedge h0 = 0$. Below we give the clues to understand this result and an outline of its proof (not found explicitly in the literature).

Definition 4.1. Let $\mathbf{A}$ be a temporal algebra. Let us denote by $O(\mathbf{A})$ the set of elements x in $\mathbf{A}$ such that $px = fx = 0$, i.e., $O(\mathbf{A}) = \{x \in A\colon px = 0 = fx\}$.

The following lemma is straightforward.

Lemma 4.1. *For any temporal algebra* $\mathbf{A}$*,* $O(\mathbf{A}) = [0, g0 \wedge h0]$ *and this set is a temporal ideal.*

Given a temporal algebra $\mathbf{A}$, in the sequel we will write F_0 (resp. F_1) instead of $[g0 \wedge h0, 1]$ (resp. $[f1 \vee p1, 1]$). Since $g_*(A) \cup h_*(A) \subseteq F_0$ it follows that $F_0 \in Tsp(\mathbf{A})$. Moreover, from the above and the obvious equality $F_1 = \neg O(\mathbf{A})$ it follows that $F_1 \in Tsp(\mathbf{A})$. The class of algebras $\mathbf{A}$ with extreme value of $O(\mathbf{A})$, namely A and $\{0\}$, provides a simple but useful theorem for our purpose. The class O (resp. W) of temporal algebras satisfying $g0 = 1$ (resp. $g0 \wedge h0 = 0$) is obviously a variety.

Definition 4.2. Given a temporal algebra $\mathbf{A}$, we define $Osp(\mathbf{A})$ and $Wsp(\mathbf{A})$ by the equalities $Osp(\mathbf{A}) = \{D \in Tsp(\mathbf{A}) : \mathbf{A}/D \in \mathsf{O}\}$ and $Wsp(\mathbf{A}) = \{D \in Tsp(\mathbf{A}) : \mathbf{A}/D \in \mathsf{W}\}$

Lemma 4.2. *For all temporal algebras* $\mathbf{A}$*,* $F_0 \in Osp(\mathbf{A})$ *and* $F_1 \in Wsp(\mathbf{A})$*.*

Proof. We have established before that $F_0, F_1 \in Tsp(\mathbf{A})$. We will prove that $\mathbf{A}/F_0 \in \mathsf{O}$. Since $fhx \leq x$, for all $x \in A$, it follows that $fh0 = 0$. Moreover, using that f is monotone we obtain that $f(h0 \wedge g0) \leq fh0$; hence $f(h0 \wedge g0) = 0$ and so $f(1/F_0) = 0/F_0$. This proves the first statement. For the second, since $(g0 \wedge h0)/F_1 = 0/F_1$, we have that $g(0/F_1) \wedge h(0/F_1) = 0/F_1$. It follows that $\mathbf{A}/F_1 \in \mathsf{W}$. ☐

Definition 4.3. In virtue of Lemma 4.2 we can define $Rad_o(\mathbf{A})$ and $Rad_w(\mathbf{A})$ by the equalities: $Rad_o(\mathbf{A}) = \bigcap Osp(\mathbf{A})$ and $Rad_w(\mathbf{A}) = \bigcap Wsp(\mathbf{A})$.

Lemma 4.3. *Let* $\mathbf{A}$ *be a temporal algebra. Then* $Rad_o(\mathbf{A}) = F_0$ *and* $Rad_w(\mathbf{A}) = F_1$*.*

Proof. It follows from Lemma 4.2 that $Rad_o(\mathbf{A}) \subseteq F_0$ and $Rad_w(\mathbf{A}) \subseteq F_1$. The converse inclusions also hold. Indeed, if $D \in Osp(\mathbf{A})$ then $g(0/D) = h(0/D) = 1/D$ and so $(g0 \wedge h0)/D = g(0/D) \wedge h(0/D) = 1/D$. This implies that $g0 \wedge h0 \in D$. Moreover, if $D \in Wsp(\mathbf{A})$ then $\mathbf{A}/D \in \mathsf{W}$. It follows that $g(0/D) \wedge h(0/D) = 0/D$, i.e., $g0 \wedge h0 \in \neg D$, or equivalently $f1 \vee p1 \in D$. ☐

Definition 4.4. Let $\mathbf{A}$ be a temporal algebra. We define $\mathbf{A}_o$ by $\mathbf{A}_o = \mathbf{A}/Rad_o(\mathbf{A})$ and $\mathbf{A}_w$ by $\mathbf{A}_w = \mathbf{A}/Rad_w(\mathbf{A})$.

Obviously, any two congruences of a Boolean algebra permute; moreover, from Lemma 4.3, $Rad_o(\mathbf{A}) \cup Rad_w(\mathbf{A}) = A$ and $Rad_o(\mathbf{A}) \cap Rad_w(\mathbf{A}) = \{1\}$. Therefore, as a consequence we have the following.

Theorem 4.4. *Let* $\mathbf{A}$ *be a temporal algebra. Then* $\mathbf{A} \cong \mathbf{A}_o \times \mathbf{A}_w$*.*

Therefore the variety T of temporal algebras is generated by the class $\mathsf{O} \cup \mathsf{W}$. If we decompose $\mathbf{F}_t(X)$ according to the previous theorem, we find that one of the factors (the one satifying $g0 \wedge h0 = 0$) has no atoms; therefore its atoms come from the other factor. So if X is finite with n elements, then we will find the form of the 2^n atoms in $\mathbf{F}_t(X)$.

Let X be a non-empty set, $\mathbf{B}(X) = \langle B(X), \wedge, \vee, \neg, 1\rangle$ the free Boolean algebra freely generated by X, and $k\colon B(X) \to B(X)$ the map defined by $k(a) = 1$ for all $a \in B(X)$. Let us denote the free temporal algebra of O (resp. W) over the set X by $\mathbf{F}_o(X)$ (resp. $\mathbf{F}_w(X)$). It is clear that $\mathbf{B}_o(X) = \langle B(X), \wedge, \vee, \neg, k, k, 1\rangle$ is in O and it coincides with $\mathbf{F}_o(X)$. This implies that the variety O is locally finite. The concept of atom in a temporal algebra depends just on the order relation of the underlying Boolean algebra. So the set of atoms of $\mathbf{B}_o(X)$ coincides with the set of atoms of $\mathbf{B}(X)$, which has 2^n elements whenever X has

n elements, i.e., if X is a non-empty set then $|Atm(\mathbf{B}_o(X))| = |Atm(\mathbf{B}(X))|$. Let $n \in \omega^*$, $\sigma \in \{-1,1\}^n$, and $X = \{x_1, \ldots, x_n\}$; define $\xi_\sigma \in B(X)$ by the equality $\xi_\sigma = \bigwedge\{x_i^{\sigma_i} : 1 \leq i \leq n\}$, where

$$\alpha^r = \begin{cases} \alpha, & \text{if } r = \ \ 1, \\ \neg\alpha, & \text{if } r = -1. \end{cases}$$

Let $n \in \omega^*$ and $X = \{x_1, \ldots, x_n\}$. Then $\mathbf{F}_o(X)$ has 2^n atoms and $Atm(\mathbf{F}_o(X)) = \{\xi_\sigma \colon \sigma \in \{0,1\}^n\}$. Moreover, note that for temporal algebras $\mathbf{A}$ and $\mathbf{B}$, $Atm(\mathbf{A} \times \mathbf{B}) = \{\langle a, 0\rangle \colon a \in Atm(\mathbf{A})\} \cup \{\langle 0, b\rangle \colon b \in Atm(\mathbf{B})\}$

To enumerate the atoms of a free temporal algebra freely generated by a finite set X we need to express $\mathbf{F}_o(X)$ and $\mathbf{F}_w(X)$ as quotients of $\mathbf{F}_t(X)$. As we will see, the suitable temporal filters are F_0 and F_1 respectively.

Theorem 4.5. *Let X be a non-empty set. Then $\mathbf{F}_o(X)$ is isomorphic to $\mathbf{F}_t(X)/F_0$ and $\mathbf{F}_w(X)$ is isomorphic to $\mathbf{F}_t(X)/F_1$.*

Proof. Let $\mathbf{A}$ be an algebra of O and let $\eta \colon X \to A$. Since $\mathbf{F}_t(X)$ is the free temporal algebra freely generated by X, there is a unique morphism of temporal algebras η' such that $\eta = \eta' \circ i$. Since $\mathbf{A}$ is in O we have $\eta'(h0 \wedge g0) = h0 \wedge g0 = 1$ and so $F_0 \subseteq ker(\eta')$. Hence there is a unique morphism $\eta'' \colon F_t(X)/F_0 \to A$ such that $\eta' = \eta'' \circ \pi$, where π is the natural morphism from $F_t(X)$ to $F_t(X)/F_0$. The first statement follows from this since the variety O is not trivial and so $\pi \circ i$ is one-to-one. Now, the rest of the proof is straightforward. ☐

Lemma 4.6. *Let $\pi(\alpha)/F_0 \in F_t(X)/F_0$. If $\pi(\alpha)/F_0 \in Atm(\mathbf{F}_t(X)/F_0)$ then $g0 \wedge h0 \wedge \pi(\alpha) \in Atm(\mathbf{F}_t(X))$.*

Proof. Let us suppose that $\pi(\alpha)/F_0$ is an atom of $F_o(X)$. If $h0 \wedge g0 \wedge \pi(\alpha) = 0$, then we would have that $0 = h0 \wedge g0/F_0 \wedge \pi(\alpha)/F_0 = 1 \wedge \pi(\alpha)/F_0 = \pi(\alpha)/F_0$ and this is impossible. Therefore $h0 \wedge g0 \wedge \pi(\alpha) \neq 0$. Let $\pi(\beta) \in F_t(X)$ be such that $\pi(\beta) \leq h0 \wedge g0 \wedge \pi(\alpha)$. Then we have the inequality $h0 \wedge g0 \wedge \pi(\beta) \leq h0 \wedge g0 \wedge \pi(\alpha)$. The above is equivalent to $\pi(\beta)/F_0 \leq \pi(\alpha)/F_0$, so either $\pi(\beta)/F_0 = 0$ or $\pi(\beta)/F_0 = \pi(\alpha)/F_0$. In the case where $\pi(\beta)/F_0 = 0$ we would have $h0 \wedge g0 \wedge \pi(\beta) = 0$ and $\pi(\beta) \leq h0 \wedge g0 \wedge \pi(\alpha) \leq h0 \wedge g0$, which implies that $\pi(\beta) = h0 \wedge g0 \wedge \pi(\beta) = 0$. In the case where $h0 \wedge g0 \wedge \pi(\alpha) = h0 \wedge g0 \wedge \pi(\beta)$, it follows that $h0 \wedge g0 \wedge \pi(\alpha) \leq \pi(\beta)$. Since we have the other inequality by choice of β, $h0 \wedge g0 \wedge \pi(\alpha) = \pi(\beta)$. ☐

Lemma 4.7. *For all $\alpha \in F(X)$, if $\pi(\alpha) \in Atm(\mathbf{F}_t(X))$ then $\pi(\alpha)/F_0 \neq 0/F_0$.*

Proof. It follows from Corollary 3.8 that $\pi(\alpha) \wedge g0 \wedge h0 = \pi(\alpha)$, whenever $\pi(\alpha) \in Atm(\mathbf{F}_t(X))$. Since $\pi(\alpha)$ is an atom, it is different of 0; therefore $\pi(\alpha) \wedge g0 \wedge h0 \neq 0$, so $\pi(\alpha)/F_0 \neq 0/F_0$. ☐

Lemma 4.8. *Let α be an element of $F(X)$. If $\pi(\alpha) \in Atm(\mathbf{F}_t(X))$ then $\pi(\alpha)/F_0 \in Atm(\mathbf{F}_t(X)/F_0)$.*

Proof. Let us assume that $\pi(\alpha) \in Atm(F_t(X))$. Lemma 4.7 ensures that $\pi(\alpha)/F_0 \neq 0/F_0$. Then suppose that $\pi(\beta)/F_0 \leq \pi(\alpha)/F_0$. It follows that $h0 \wedge g0 \wedge \pi(\beta) \leq \pi(\alpha)$ and this implies that either $h0 \wedge g0 \wedge \pi(\beta) = 0$ or $h0 \wedge g0 \wedge \pi(\beta) = \pi(\alpha)$. The first equality is equivalent to $\pi(\beta)/F_0 = 0/F_0$ and the second one to $\pi(\beta)/F_0 = \pi(\alpha)/F_0$. ☐

Theorem 4.9. *Let X be an infinite set. Then $\mathbf{F}_t(X)$ is atomless.*

Proof. $|Atm(\mathbf{B}_o(X))|$ coincides with $|Atm(\mathbf{B}(X))|$ and so $\mathbf{B}_o(X)$ is atomless; hence $\mathbf{F}_t(X)/F_0$ is atomless. Let us assume that $\pi(\alpha) \in F_t(X)$ is an atom. By Lemma 4.8, $\pi(\alpha)/F_0$ is an atom of $\mathbf{F}_t(X)/F_0$. Since this is impossible, $\mathbf{F}_t(X)$ has no atom. ☐

Remark 4.1. Let $\pi(\alpha)/F_0, \pi(\beta)/F_0 \in F_t(X)/F_0$. It is clear that $\pi(\alpha)/F_0 \neq \pi(\beta)/F_0$ if, and only if, $h0 \wedge g0 \wedge \pi(\alpha) \neq h0 \wedge g0 \wedge \pi(\beta)$.

Theorem 4.10. *Let $n \in \omega^*$ and $X = \{x_1, \ldots, x_n\}$ be a finite set with cardinality n. $\mathbf{F}_t(X)$ has 2^n atoms and $Atm(\mathbf{F}_t(X)) = \{\xi_\sigma \wedge h0 \wedge g0 \colon \sigma \in \{0,1\}^n\}$.*

Proof. Let us consider the mapping $\Psi\colon Atm\big(\mathbf{F}_t(X)/F_0\big) \to Atm\big(\mathbf{F}_t(X)\big)$ given by $\Psi(\alpha/F_0) = h0 \wedge g0 \wedge \pi(\alpha)$, for all $\pi(\alpha)/F_0 \in Atm\big(\mathbf{F}_t(X)/F_0\big)$. This map is well-defined as guaranteed by Remark 4.1 and Lemma 4.6. Now, let us consider the mapping $\Upsilon\colon Atm\big(\mathbf{F}_t(X)\big) \to Atm\big(\mathbf{F}_t(X)/F_0\big)$ given by $\Upsilon\big(\pi(\alpha)\big) = \pi(\alpha)/F_0$, which is also well-defined as indicates Lemma 4.8. On one hand

$$\begin{aligned}(\Upsilon \circ \Psi)(\pi(\alpha)/F_0) &= \Upsilon\big(h0 \wedge g0 \wedge \pi(\alpha)\big) = \big(h0 \wedge g0 \wedge \pi(\alpha)\big)/F_0 \\ &= (h0 \wedge g0)/F_0 \wedge \pi(\alpha)/F_0 = 1/F_0 \wedge \pi(\alpha)/F_0 \\ &= \pi(\alpha)/F_0\end{aligned}$$

and on the other hand $(\Psi \circ \Upsilon)(\pi(\alpha)) = \Psi(\pi(\alpha)/F_0) = h0 \wedge g0 \wedge \pi(\alpha)$. Since $0 \leq h0 \wedge g0 \wedge \pi(\alpha) \leq \pi(\alpha)$ and $\pi(\alpha) \in Atm\big(\mathbf{F}_t(X)\big)$, then either $h0 \wedge g0 \wedge \pi(\alpha) = 0$ or $h0 \wedge g0 \wedge \pi(\alpha) = \pi(\alpha)$. The first case is impossible since then we would have $\pi(\alpha)/F_0 = h0 \wedge g0 \wedge \pi(\alpha)/F_0 = 0/F_0$ and so $\pi(\alpha)/F_0$ would not be in $Atm\big(\mathbf{F}_t(X)/F_0\big)$, in contradiction with Lemma 4.8. Therefore $h0 \wedge g0 \wedge \pi(\alpha) = \pi(\alpha)$ and so $(\Psi \circ \Upsilon)(\pi(\alpha)) = \pi(\alpha)$, for all $\pi(\alpha) \in Atm\big(\mathbf{F}_t(X)\big)$. It follows that Ψ and Υ are mutually inverse and bijective mappings. Hence the cardinality of $Atm\big(\mathbf{F}_t(X)\big)$ coincides with the cardinality of $Atm\big(\mathbf{F}_t(X)/F_0\big)$. From the definition of Ψ, it follows that $Atm\big(\mathbf{F}_t(X)\big) = \{\xi_\sigma \wedge h0 \wedge g0 \colon \sigma \in \{-1,1\}^n\}$. □

Finally we will study whether the algebra $\mathbf{F}_t(X)$, in case X is finite, is atomic or not.

Theorem 4.11. *Let X be a non-empty set. The algebra $\mathbf{F}_w(X)$ is atomless.*

Proof. Let us assume first that X is infinite. Using Theorem 4.4 we have $\mathbf{F}_t(X) \cong \mathbf{F}_t(X)/F_0 \times \mathbf{F}_t(X)/F_1$. Since $\mathbf{F}_t(X)$ is atomless it follows that $\mathbf{F}_t(X)/F_1$ is also atomless. But, after Theorem 4.5, $\mathbf{F}_w(X) \cong \mathbf{F}_t(X)/F_1$ and so $\mathbf{F}_w(X)$ is atomless. Let us suppose now that X is finite. Using again Theorem 4.4 it follows that $|Atm\big(\mathbf{F}_t(X)\big)| = |Atm\big(\mathbf{F}_o(X)\big)| + |Atm\big(\mathbf{F}_w(X)\big)|$. By Theorem 4.10, $|Atm\big(\mathbf{F}_t(X)\big)| = |Atm\big(\mathbf{F}_o(X)\big)|$ and so $\mathbf{F}_w(X)$ is atomless. □

Corollary 4.12. *Let X be a finite set. The algebra $\mathbf{F}_t(X)$ is not atomic.*

Proof. Let us consider the temporal algebra $\mathbf{F}_o(X) \times \mathbf{F}_w(X)$ and let us take $a \in F_w(X) \setminus \{0\}$. If $\mathbf{F}_o(X) \times \mathbf{F}_w(X)$ were to be atomic, then it would exist an atom $\langle x, y\rangle$ such that $\langle x, y\rangle \leq \langle 0, a\rangle$. So we would have $x = 0$ and $y \in Atm\big(\mathbf{F}_w(X)\big)$, which is impossible since $\mathbf{F}_w(X)$ is atomless. Therefore $\mathbf{F}_o(X) \times \mathbf{F}_w(X)$ is not atomic and so $\mathbf{F}_t(X)$ is not atomic. □

Acknowledgements

This work is supported by grant MTM2004-03101 of Spanish Ministerio de Educación y Ciencia, which includes FEDER funds from European Union. We are indebted to the director of this project, Josep Maria Font, for his many suggestions and corrections to this work.

References

[1] Bellissima, F. Atoms in modal algebras. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 30:303–312, 1984.

[2] Bellissima, F. Atoms of tense algebras. *Algebra Universalis*, 28:52–78, 1991.

[3] Burris, S. and Sankappanavar, H.P. *A Course in Universal Algebra.* Springer-Verlag, 1981.

[4] García Olmedo, F.M. and Rodríguez Salas, A. J. A structure theorem for free temporal algebras. *Mathematical Logic Quartely*, 41:249–256, 1995.

[5] García Olmedo, F.M. and Rodríguez Salas, A. J. Temporal algebras, pretemporal algebras, and modal algebras. A relation between time and necessity. *Mathematical Logic Quartely*, 41:24–38, 1995.

[6] Kracht, M. Even more about the lattice of tense logics. *Archive for Math. Logic*, 31:243–257, 1992.

[7] Wolter, F. A note on atoms in polymodal algebras. *Algebra Universalis*, 37:334–341, 1997.