DISTRIBUTIVE LATTICES WITH A GENERALIZED IMPLICATION: TOPOLOGICAL DUALITY

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Abstract. In this paper we introduce the notion of *generalized implication* for lattices, as a binary function $\Rightarrow$ that maps every pair of elements of a lattice to an ideal. We prove that a bounded lattice A is distributive if and only if there exists a generalized implication $\Rightarrow$ defined in A satisfying certain conditions, and we study the class of bounded distributive lattices A endowed with a generalized implication as a common abstraction of the notions of annihilator [6], Quasi-modal algebras [3], and weakly Heyting algebras [5]. We introduce the suitable notions of morphisms in order to obtain a category, as well as the corresponding notion of congruence. We develop a Priestley style topological duality for the bounded distributive lattices with a generalized implication. This duality generalizes the duality given in [5] for weakly Heyting algebras and the duality given in [3] for Quasi-modal algebras.

1. Introduction

The purpose of this paper is to introduce and study the class of bounded distributive lattices endowed with a binary function $\Rightarrow$, called *generalized implication*, as a common abstraction of the notion of annihilator [6], quasi-modal algebras [3], and weakly Heyting algebras [5]. A generalized implication on a bounded distributive lattice A is a binary function that maps every pair of elements (a, b) of A to an ideal $a \Rightarrow b$ of A and satisfies for every $a, b, c \in A$:

1. $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \wedge c)$,
2. $(a \Rightarrow c) \cap (b \Rightarrow c) = (a \vee b) \Rightarrow c$,
3. $(a \Rightarrow b) \cap (b \Rightarrow c) \subseteq a \Rightarrow c$,
4. $a \Rightarrow a = A$.

A pair $\langle A, \Rightarrow \rangle$ where A is a bounded distributive lattice and $\Rightarrow$ is a generalized implication will be called a *distributive lattice with a generalized implication*, or *gi-lattice* for short. A structure of this shape is not an algebra in the sense considered in Universal Algebra, where an algebra is a set plus a collection of operations on it. Nonetheless a generalized implication has similar properties to the implication of weakly Heyting algebras.

The paper is organized in the following fashion. In Section 2 we give some necessary notations and definitions, and recall Priestley duality for bounded distributive lattices. Section 3 gives the definition of gi-lattice and proves some of its properties. In that section we also present some examples which motivate the study of the class of gi-lattices, and we see that the notion of annihilator is a particular case of the notion of generalized implication on bounded distributive lattices. We prove that a bounded lattice A is distributive if and only if there exists a generalized implication $\Rightarrow$ defined in A satisfying certain conditions (see

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Theorem 3). We also introduce a notion of homomorphism between gi-lattices (gi-homomorphisms) and the corresponding notion of congruence (gi-congruences). In Section 4 we prove a representation theorem for gi-lattices. In Section 5 the representation is turned into a topological Priestley duality between the category of gi-lattices with gi-homomorphisms as arrows and certain relational Priestley spaces with gi-morphism as arrows. These results extend the duality developed in [5] for weakly Heyting algebras, and the duality given in [3] for Quasi-modal algebras. Finally, in Section 6 we first apply the duality to characterize the closed sets that correspond to gi-congruences and then, using the fact that lattice filters correspond to closed up-sets of the dual space, we characterize the filters that correspond to closed up-sets whose dual congruence is a gi-congruence. We call these filters gi-filters. To conclude we characterize the lattice of gi-congruences isomorphic to the lattice of gi-filters.

2. Preliminaries

We adopt the following standard conventions. If X is a set and R a binary relation on X, then for every $x \in X$, $R(x)$ denotes the image of x by R. More precisely, $R(x) = \{y \in X : (x, y) \in R\}$. Also, if $f : X \to Y$ is a function and $Z \subseteq X$ then $f[Z]$ denotes the image of Z by f, i.e., $f[Z] = \{f(z) : z \in Z\}$. If $Y \subseteq X$, then Y^c denotes the set-theoretical complement of Y relative to X, i.e. $Y^c = X \smallsetminus Y$.

The category of bounded distributive lattices with their homomorphisms will be denoted by **DLat** and the class of its objects by DLat. If $A \in \mathsf{DLat}$ then $\mathrm{Fi}(A)$ and $\mathrm{Id}(A)$ will respectively denote the family of the filters of A and the family of ideals of A. The filter (ideal) generated by a subset $X \subseteq A$ will be denoted by $\mathrm{F}(X)$ $(\mathrm{I}(X))$. We will write $\uparrow a$ $(\downarrow a)$ to refer to the filter (ideal) generated by $\{a\}$. The family of the prime filters of A is denoted by $X(A)$. Given $A \in \mathsf{DLat}$, let $\beta : A \to \mathcal{P}(X(A))$ be the Stone map defined by

$$\beta(a) = \{P \in X(A) : a \in P\},$$

for every $a \in A$. The family $\beta[A]$ is closed under unions, intersections, and contains $\emptyset$ and A; it is therefore a bounded distributive lattice and we denote it by $\beta(A)$. It is well known that the map β establishes an isomorphism between A and $\beta(A)$.

A *totally order-disconnected topological space* is a triple $X = \langle X, \leq, \tau_X \rangle$, where $\langle X, \leq \rangle$ is a poset, $\langle X, \tau_X \rangle$ is a topological space and given $x, y \in X$ such that $x \nleq y$ there is a clopen up-set U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. A morphism between Priestley spaces is a continuous and monotone function between them. The category of Priestley spaces together with their morphisms will be denoted by **Pr** and the class of its objects by Pr. If $X \in \mathsf{Pr}$ the family of all clopen up-sets of X is denoted by $CU(X)$, and it is well known that $CU(X) = \langle CU(X), \cup, \cap, \emptyset, X \rangle$ is a bounded distributive lattice. The family of open up-sets of X will be denoted by $\mathcal{O}_u(X)$ and the family of closed up-sets by $\mathcal{C}_u(X)$.

The Priestley space of a bounded distributive lattice A is the triple $X(A) = \langle X(A), \subseteq, \tau_{X(A)} \rangle$, where $\tau_{X(A)}$ is the topology generated by taking as a subbase the family $\{\beta(a) : a \in P\} \cup \{\beta(a)^c : a \in P\}$, where $\beta(a)^c = X(A) \smallsetminus \beta(a)$. In this case, we have $A \cong CU(X(A))$.

If $X \in \mathsf{Pr}$, then the map $\varepsilon_X : X \to X(CU(X))$ defined by

$$\varepsilon_X(x) = \{U \in CU(X) : x \in U\},$$

for every $x \in X$, is a homeomorphism and an order-isomorphism.

Let $A_1, A_2 \in \mathsf{DLat}$ and let $h : A_1 \to A_2$ be a homomorphism. The map $h_* : X(A_2) \to X(A_1)$ defined by $h_*(P) = h^{-1}(P)$, for each $P \in X(A_2)$, is a continuous and monotone function. If X and Y are Priestley spaces and $f : X \to Y$ is a continuous and monotone function, then the map $f^* : CU(Y) \to CU(X)$ defined by $f^*(U) = f^{-1}(U)$, for each $U \in CU(Y)$, is a bounded lattice homomorphism.

Let $(.)^* : \mathbf{Pr} \to \mathbf{DLat}$ be the contravariant functor defined as follows: for every $X \in \mathsf{Pr}$, $(X)^* = CU(X)$, and for every morphism $f : X \to Y$ with $X, Y \in \mathsf{Pr}$, $(f)^* = f^*$. And let $(-)_* : \mathbf{DLat} \to \mathbf{Pr}$ be the contravariant functor such that for every $A \in \mathsf{DLat}$ $(A)_* = X(A)$ and for every homomorphism $h : A_1 \to A_2$ with $A_1, A_2 \in \mathsf{DLat}$ $(h)_* = h_*$. These functors establish a dual equivalence between the categories **DLat** and **Pr**.

3. Distributive lattices with a generalized implication

Definition 1. Let A be a lattice. A *generalized implication* on A is a function $\Rightarrow : A \times A \to \mathrm{Id}(A)$ such that for every $a, b, c \in A$:

G1. $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \wedge c)$,
G2. $(a \Rightarrow c) \cap (b \Rightarrow c) = (a \vee b) \Rightarrow c$,
G3. $(a \Rightarrow b) \cap (b \Rightarrow c) \subseteq a \Rightarrow c$,
G4. $a \Rightarrow a = A$.

A *distributive lattice with a generalized implication,* or gi-lattice for short, is a pair $\mathbf{A} = \langle A, \Rightarrow \rangle$, where A is a bounded distributive lattice and $\Rightarrow$ is a generalized implication defined on A. We denote the class of bounded distributive lattices with a generalized implication, or gi-lattices, by DLatGi.

Proposition 2. *Let A be a lattice and $\Rightarrow$ a generalized implication on A. Then the following conditions hold,*

1. *if $a \leq b$ then $(c \Rightarrow a) \subseteq (c \Rightarrow b)$ and $(b \Rightarrow c) \subseteq (a \Rightarrow c)$,*
2. *if $a \leq b$ then $a \Rightarrow b = A$,*
3. *$(a \Rightarrow b) \cap (a \Rightarrow c) \subseteq a \Rightarrow (b \vee c)$.*

Proof. (1) If $a \leq b$, then $a \vee b = b$ and $a \wedge b = a$. Using G2, we obtain $(b \Rightarrow c) = (a \Rightarrow c) \cap (b \Rightarrow c)$, and therefore, $(b \Rightarrow c) \subseteq (a \Rightarrow c)$. Using G1 we obtain $(c \Rightarrow a) = (c \Rightarrow a) \cap (c \Rightarrow b)$, and consequently $(c \Rightarrow a) \subseteq (c \Rightarrow b)$.

(2) follows from (1) using G4, and (3) is a consequence of the fact that $(b \wedge c) \leq (b \vee c)$, (1) and G1. □

We now discuss some examples of lattices with a generalized implication.

Example 1. Let $\langle X, \leq, R \rangle$ be a relational structure where $\langle X, \leq \rangle$ is a poset and R is a binary relation on X such that $(\leq \circ R) \subseteq R$. Let $\to$ be the binary operation on the set $\mathcal{P}_u(X)$ of up-sets of $\langle X, \leq \rangle$ defined by

$$U \to_R V = \{x \in X : R(x) \cap U \subseteq V\},$$

for every $U, V \in \mathcal{P}_u(X)$. Condition $(\leq \circ R) \subseteq R$ implies that $U \to_R V \in \mathcal{P}_u(X)$. Let for every $U, V \in \mathcal{P}_u(X)$

$$U \Rightarrow_R V = \{W \in \mathcal{P}_u(X) : W \subseteq (U \to_R V)\},$$

that is, $U \Rightarrow_R V$ is the ideal generated by $U \to_R V$. Then $\langle \mathcal{P}_u(X), U \Rightarrow_R V \rangle$ is a gi-lattice. The proof of this fact is as follows: First of all notice that by definition $U \Rightarrow_R V$ is an ideal of $\mathcal{P}_u(X)$. Now we show that the conditions of Definition 1 hold. Let $U, V, T \in \mathcal{P}_u(X)$. To prove that condition G1 holds, assume that $W \in (U \Rightarrow_R V) \cap (U \Rightarrow_R T)$. Thus, $R(x) \cap U \subseteq V$ and $R(x) \cap U \subseteq T$, for every $x \in W$. Therefore $R(x) \cap U \subseteq V \cap T$. It follows that $W \in (U \Rightarrow_R (V \cap T))$. Hence $(U \Rightarrow_R V) \cap (U \Rightarrow_R T) \subseteq (U \Rightarrow_R (V \cap T))$. The other inclusion is proved similarly. In order to prove condition G2, let $W \in (U \Rightarrow_R V) \cap (T \Rightarrow_R V)$. Thus, for every $x \in W$ we have $R(x) \cap U \subseteq V$ and $R(x) \cap T \subseteq V$. Hence, $R(x) \cap (U \cup T) \subseteq V$.

This implies that $W \in (U \cup T) \Rightarrow_R V$. The other inclusion follows immediately using distributivity. Now we prove condition G3. Let $W \in (U \Rightarrow_R V) \cap (V \Rightarrow_R T)$. Therefore, for every $x \in W$ we have $R(x) \cap U \subseteq V$ and $R(x) \cap V \subseteq T$. Thus, $R(x) \cap U \subseteq T$. Hence $W \in U \Rightarrow_R T$. Finally we prove that condition G4 holds. For every $x \in X$, $R(x) \cap U \subseteq U$. Therefore, $(U \to_R U) = X$. So, $U \Rightarrow_R U = \mathcal{P}_u(X)$.

Example 2. Let $\langle X, \leq, R\rangle$ be, as in the previous example, a relational structure where $\langle X, \leq\rangle$ is a poset and R is a binary relation on X such that $(\leq \circ R) \subseteq R$. Let D be a bounded sublattice of $\mathcal{P}_u(X)$ and let us define $\Rightarrow_R^D$ by

$$U \Rightarrow_R^D V = \{W \in D : W \subseteq (U \to_R V)\}.$$

This set is clearly an ideal of D. Now if we replace $\mathcal{P}_u(X)$ by D everywhere in the proof given in Example 1, we obtain a proof of the fact that $\langle D, \Rightarrow_R^D\rangle$ is a gi-lattice.

Example 3. In every bounded distributive lattice A it is possible to define a generalized implication $\Rightarrow$ by

$$a \Rightarrow b = \begin{cases} A \text{ if } a \leq b \\ \{0\} \text{ if } a \nleq b \end{cases}.$$

It is easy to see that $\langle A, \Rightarrow\rangle$ is a gi-lattice.

Example 4. Let ω be the set of the natural numbers. For every $U, V \subseteq \omega$ let

$$U \Rightarrow V = \begin{cases} \{Y \subseteq \omega : Y \subseteq U^c \cup V \text{ and } Y \text{ is finite}\}, & \text{if } U^c \cup V \neq \omega \\ \\ \mathcal{P}(\omega), & \text{otherwise} \end{cases}$$

It is easy to check that $U \Rightarrow V$ is an ideal of $\mathcal{P}(\omega)$. Thus, the structure

$$\langle \mathcal{P}(\omega), \cup, \cap, \Rightarrow, \emptyset, \omega\rangle$$

is a gi-lattice.

Example 5. In [3] and [2], the notions of Quasi-Modal algebras and Quasi-Modal lattice are introduced, respectively. A quasi-modal operator on a Boolean algebra B is a map $\Delta : B \to \mathrm{Id}(B)$ such that $\Delta 1 = B$ and $\Delta(a \wedge b) = \Delta a \cap \Delta b$ for every $a, b \in B$. If $\Rightarrow$ is a generalized implication on a Boolean algebra B, then the unary operation Δ defined by $\Delta a := 1 \Rightarrow a$ is a quasi-modal operator on B. Conversely, if Δ is a quasi-modal operator on B, then the map $\Rightarrow : B \times B \to \mathrm{Id}(B)$ defined $a \Rightarrow b := \Delta(-a \vee b)$ is a generalized implication. Thus, the gi-lattices $\langle B, \Rightarrow\rangle$, where B is a Boolean algebra, are interdefinable with the Quasi-Modal algebras defined in [3].

Example 6. A distributive lattice with a generalized implication can also be seen as an extension of the notion of weakly Heyting algebra introduced in [5]. The variety of weakly Heyting algebras is the algebraic counterpart of the least subintuitionistic logic wK considered in [4]. Let us recall that a *weakly Heyting algebra,* or WH-algebra, is a pair $\langle A, \to\rangle$, where A is a bounded distributive lattice and $\to : A \times A \to A$ is a map such that for all $x, y, z \in A$,

(1) $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$,
(2) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$,
(3) $(x \to y) \wedge (y \to z) \leq x \to z$,
(4) $x \to x = 1$.

Then the binary map $\Rightarrow : A \times A \to \mathrm{Id}(A)$ defined by setting $a \Rightarrow b := I(a \to b)$ is a generalized implication on A. Conversely, if $\Rightarrow$ is a generalized implication on a bounded distributive lattice A with the property that for every $a, b \in A$, $a \Rightarrow b$ is a principal ideal, then the pair $\langle A, \to\rangle$ where $\to : A \times A \to A$ is defined by $a \to b =$ the x such that $a \Rightarrow b = {\downarrow}x$, is a weakly Heyting algebra.

Example 7. It is well known that in a lattice A the *annihilator* $\langle a, b\rangle$ of a relative to b is defined by $\langle a, b\rangle = \{x \in A : x \wedge a \leq b\}$. Several authors have studied annihilators in lattices. In particular, Mandelker [6] proved that a lattice A is distributive if and only if $\langle a, b\rangle$ is an ideal for all $a, b \in A$. The concept of annihilator is a natural generalization of the relative pseudocomplement $a \to b$ of an element $a \in A$ relative to an element $b \in A$. It is clear that if A is a bounded distributive lattice, then the function $\langle\ \rangle : A \times A \to \mathrm{Id}(A)$ defined by $\langle\ \rangle(a, b) = \langle a, b\rangle$, for each $a, b \in A$ satisfies the conditions G1-G4 of Definition 1. Thus the notion of generalized implication is an extension of the concept of annihilator in distributive lattices. Now we prove that the property of distributivity in a bounded lattice can be characterized in terms of the existence of a particular generalized implication.

Theorem 3. *Let A be a bounded lattice. Then A is distributive if and only if there exists a generalized implication $\Rightarrow$ on A satisfying the conditions*

1. *$a \in 1 \Rightarrow a$,*
2. *${\downarrow}a \cap (a \Rightarrow b) \subseteq {\downarrow}b$, for all $a, b \in A$.*

Proof. $\Rightarrow$) As A is distributive, the annihilator $\langle a, b\rangle$ is an ideal for each $a, b \in A$. By the definition of annihilator it is clear that $a \in \langle 1, a\rangle$, and ${\downarrow}a \cap \langle a, b\rangle \subseteq {\downarrow}b$, for all $a, b \in A$. Thus, there exists a generalized implication, namely $\langle\ \rangle$ on A, satisfying (1) and (2).

$\Leftarrow$) If we prove that $\langle a, b\rangle = a \Rightarrow b$, for all $a, b \in A$, then by the results of [6], we have that A is distributive. Let $a, b \in A$. We first prove that $\langle a, b\rangle \subseteq a \Rightarrow b$. Let $x \in \langle a, b\rangle$, i.e. $x \wedge a \leq b$. Then $a \Rightarrow (x \wedge a) \subseteq a \Rightarrow b$. Hence,

$$a \Rightarrow x = (a \Rightarrow x) \cap A = (a \Rightarrow x) \cap (a \Rightarrow a) = a \Rightarrow (x \wedge a) \subseteq a \Rightarrow b.$$

As $a \leq 1$, we obtain $1 \Rightarrow x \subseteq a \Rightarrow x$. Thus, $x \in 1 \Rightarrow x \subseteq a \Rightarrow x \subseteq a \Rightarrow b$. In order to prove the other inclusion, let $x \in a \Rightarrow b$. Then ${\downarrow}x \subseteq a \Rightarrow b$. So,

$${\downarrow}x \cap {\downarrow}a = {\downarrow}(x \wedge a) \subseteq {\downarrow}a \cap (a \Rightarrow b) \subseteq {\downarrow}b.$$

Thus, $x \wedge a \leq b$ and so $x \in \langle a, b\rangle$. □

3.1. Morphisms between distributive lattices with a generalized implication.

We introduce the notion of lattice homomorphism that preserves the generalized implication.

Definition 4. Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathsf{DLatGi}$. A bounded lattice homomorphism $h : A_1 \to A_2$ is a *gi-homomorphism* from $\mathbf{A}_1$ to $\mathbf{A}_2$ if

$$I(h\,[a \Rightarrow_1 b]) = (h(a) \Rightarrow_2 h(b)),$$

for every $a, b \in A_1$. If in addition h is a lattice isomorphism, then h is called a *gi-isomorphism* and in this situation we write $\mathbf{A}_1 \underset{g}{\cong} \mathbf{A}_2$.

The set of gi-homomorphisms from $\mathbf{A}_1$ to $\mathbf{A}_2$ contains the identity function and it is closed under composition, as we see in the lemma below. Thus the class of gi-lattices, taken as objects, and their gi-homomorphisms, taken as arrows, form a category that we denote by **DLatGi**.

Lemma 5. *The composition of gi-homomorphisms is a gi-homomorphism.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \in \mathsf{DLatGi}$, and let $f : A_1 \to A_2$ and $g : A_2 \to A_3$ be two gi-homomorphisms. We show that $g \circ f$ is a gi-homomorphism. We need to prove that for every $a, b \in A_1$

$$I((g \circ f)\,[a \Rightarrow_1 b]) = g(f(a)) \Rightarrow_3 g(f(b)). \tag{3.1}$$

In order to prove this identity, it is enough to show that

$$I(g\,[f\,[a \Rightarrow_1 b]]) = I(g\,[I(f\,[a \Rightarrow_1 b])]) \tag{3.2}$$

because

$$\begin{aligned} I(g\,[f\,[a \Rightarrow_1 b]]) &= I(g\,[I(f\,[a \Rightarrow_1 b])]) = I(g\,[f(a) \Rightarrow_2 f(b)]) \\ &= g(f(a)) \Rightarrow_3 g(f(b)). \end{aligned}$$

Let us prove (3.2). Since $g\,[f\,[a \Rightarrow_1 b]] \subseteq I(g\,[f\,[a \Rightarrow_1 b]])$,

$$I(g\,[f\,[a \Rightarrow_1 b]]) \subseteq I(g\,[I(f\,[a \Rightarrow_1 b])]).$$

To prove the other inclusion, let $x \in I(g\,[I(f\,[a \Rightarrow_1 b])])$. Hence, there are $z_1, \dots, z_n \in g\,[I(f\,[a \Rightarrow_1 b])]$ such that $x \leq z_1 \vee \dots \vee z_n$. Let $y_1, \dots, y_n \in I(f\,[a \Rightarrow_1 b])$ such that $z_1 = g(y_1), \dots, z_n = g(y_n)$. Then $x \leq g(y_1) \vee \dots \vee g(y_n) = g(y_1 \vee \dots \vee y_n)$. As $y_1 \vee \dots \vee y_n \in I(f\,[a \Rightarrow_1 b])$, there are $d_1, \dots, d_k \in a \Rightarrow_1 b$ such that

$$y_1 \vee \dots. \vee y_n \leq f(d_1) \vee \dots \vee f(d_k) = f(d_1 \vee \dots \vee d_k).$$

Thus, $x \leq g(f(d_1 \vee \dots \vee d_k))$ and since $d_1 \vee \dots \vee d_k \in (a \Rightarrow_1 b)$, we obtain $x \in I(g\,[f\,[a \Rightarrow_1 b]])$. ☐

3.2. **gi-congruences.** The notion of gi-homomorphism leads naturally to a notion of congruence for gi-lattices. We establish the relevant properties of the kernel of a gi-homomorphism.

Proposition 6. *Let* $\mathbf{A}_1, \mathbf{A}_2 \in \mathsf{DLatGi}$ *and* $h : \mathbf{A}_1 \to \mathbf{A}_2$ *a gi-homomorphism. Then* $\ker h = \{\langle a, b\rangle : h(a) = h(b)\}$ *is a congruence of* A_1 *and for every* $(a, b), (c, d) \in \ker h$,

(1) $\forall x \in (a \Rightarrow_1 c)\ \exists y \in (b \Rightarrow_1 d) : (x, y) \in \ker h$,
(2) $\forall x \in (b \Rightarrow_1 d)\ \exists y \in (a \Rightarrow_1 c) : (x, y) \in \ker h$.

Proof. It is well known that $\ker h$ is a congruence of A_1. Suppose that $(a, b), (c, d) \in \ker h$. So, $h(a) = h(b)$ and $h(c) = h(d)$. To prove (1) let $x \in (a \Rightarrow_1 c)$. Then $h(x) \in h[a \Rightarrow_1 c] \subseteq I(h[a \Rightarrow_1 c])$. Since h is a gi-homomorphism, $h(x) \in (h(a) \Rightarrow_2 h(c)$. Therefore, $h(x) \in (h(b) \Rightarrow_2 h(d) = I(h[b \Rightarrow_1 d])$. Since $b \Rightarrow_1 d$ is an ideal, there must be $y \in b \Rightarrow_1 d$ such that $h(x) \leq h(y)$. Then $h(x) = h(x \wedge y)$ and $x \wedge y \in b \Rightarrow_1 d$. Hence $(x, x \wedge y) \in \ker h$. So, (1) holds. (2) is proved similarly. ☐

Proposition 6 suggests the following definition.

Definition 7. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ and θ a congruence of $\mathbf{A}$. We say that θ is a *gi-congruence* of $\mathbf{A}$ if for every $(a, b), (c, d) \in \theta$,

(1) $\forall x \in (a \Rightarrow c)\ \exists y \in (b \Rightarrow d) : (x, y) \in \theta$,
(2) $\forall x \in (b \Rightarrow d)\ \exists y \in (a \Rightarrow c) : (x, y) \in \theta$.

Let θ be a congruence of the bounded distributive lattice A of $\mathbf{A} \in \mathsf{DLatGi}$. To indicate that $(a, b), (c, d) \in \theta$ satisfy conditions (1) and (2) of the above definition, we will use the expression

$$\langle a \Rightarrow c, b \Rightarrow d\rangle \in \theta.$$

The correspondence between homomorphism and congruences of a bounded distributive lattice specializes to a correspondence between gi-homomorphisms and gi-congruences for gi-lattices.

Let $\mathbf{A} \in \mathsf{DLatGi}$ and θ a gi-congruence of $\mathbf{A}$. We define the quotient $\mathbf{A}/\theta$ as the pair $\langle A/\theta, \Rightarrow_\theta\rangle$ where A/θ is the quotient lattice of A by θ and $\Rightarrow_\theta$ is defined as follows:

$$[a] \Rightarrow_\theta [b] = \{[c] : c \in a \Rightarrow b\}$$

for every $a, b \in A$. Conditions (1) and (2) of Definition 7 guarantee that the definition of $\Rightarrow_\theta$ is independent of the choice of the representatives of the equivalence classes. Let $\pi_\theta : A \to A/\theta$ be the canonical homomorphism (i.e. $\pi_\theta(a) = [a]$). So, $[a] \Rightarrow_\theta [b] = \pi_\theta[a \Rightarrow b]$. Since A is a bounded distributive lattice, A/θ is a bounded distributive lattice. And since for every ideal I of A, $\pi_\theta[I]$ is an ideal of A/θ, it follows that for every $a, b \in A$, $\pi_\theta[a \Rightarrow b]$ is an ideal of A/θ. Therefore, $[a] \Rightarrow_\theta [b]$ is an ideal. Hence:

Lemma 8. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and* θ *a gi-congruence of* $\mathbf{A}$. *Then* π_θ *is such that*

$$I(\pi_\theta[a \Rightarrow b]) = \pi_\theta(a) \Rightarrow_\theta \pi_\theta(b)$$

for every $a, b \in A$.

Proposition 9. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and* θ *a gi-congruence of* $\mathbf{A}$. *Then* $\mathbf{A}/\theta$ *is a gi-lattice and* π_θ *a gi-homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}/\theta$.

Proof. First we show that $\mathbf{A}/\theta$ is a gi-lattice. Since A is also a bounded distributive lattice, A/θ is a bounded distributive lattice. The lemma above gives that for every $a, b \in A$, $[a] \Rightarrow_\theta [b]$ is an ideal of $\mathbf{A}/\theta$. Now we show that conditions G1-G4 of Definition 1 hold. Let $a, b, c \in A$. Suppose that $[d] \in ([a] \Rightarrow_\theta [b]) \cap ([a] \Rightarrow_\theta [c])$. Let $d' \in a \Rightarrow b$ and $d'' \in a \Rightarrow c$ such that $d\theta d'$ and $d\theta d''$. Then $d' \wedge d'' \in (a \Rightarrow b) \cap (a \Rightarrow c)$. Therefore, by condition G1, $d' \wedge d'' \in (a \Rightarrow (b \wedge c)$ and so $[d' \wedge d''] \in [a] \Rightarrow_\theta ([b] \wedge [c])$. Now notice that $[d' \wedge d''] = [d'] \wedge [d''] = [d]$. Therefore, $[d] \in [a] \Rightarrow_\theta ([b] \wedge [c])$. This proves that $([a] \Rightarrow_\theta [b]) \cap ([a] \Rightarrow_\theta [c]) \subseteq [a] \Rightarrow_\theta ([b] \wedge [c])$. The other inclusion is proved similarly. Hence we obtain condition G1. In a similar manner we prove that conditions G2 and G3 hold. Finally, from the fact that $a \Rightarrow a = A$, condition G4 follows.

From Lemma 8 it follows that π_θ is a gi-homomorphism. □

Proposition 10. *Let* $\mathbf{A}_1, \mathbf{A}_2 \in \mathsf{DLatGi}$ *and* $h : \mathbf{A}_1 \to \mathbf{A}_2$ *an onto gi-homomorphism. Then there exists a gi-isomorphism* $f : \mathbf{A}_1/\mathrm{ker}h \to \mathbf{A}_2$ *such that* $h = f \circ \pi_{\mathrm{ker}h}$.

Proof. Since $\mathrm{ker}h$ is a gi-congruence, the quotient $\mathbf{A}_1/\mathrm{ker}h$ is well defined. By general results of Universal Algebra we know that there is an isomorphism $f : A_1/\mathrm{ker}h \to A_2$ such that $h = f \circ \pi_{\mathrm{ker}h}$. We show that f is a gi-isomorphism. We have to prove that for every $a, b \in A_1$, $I(f[[a] \Rightarrow_{\mathrm{ker}h} [b]]) = f(a) \Rightarrow_2 f(b)$. Since $[a] \Rightarrow_{\mathrm{ker}h} [b]$ is an ideal of $A_1/\mathrm{ker}h$ and f is an isomorphism, $f[[a] \Rightarrow_{\mathrm{ker}h} [b]]$ is an ideal of A_2. So, we have to show that $f[[a] \Rightarrow_{\mathrm{ker}h} [b]] = f(a) \Rightarrow_2 f(b)$. Using that $\pi_{\mathrm{ker}h}$ is a gi-homomorphism and that $\pi_{\mathrm{ker}h}[a \Rightarrow_1 b]$ is an ideal of $A_1/\mathrm{ker}h$ we have:

$$f[[a] \Rightarrow_{\mathrm{ker}h} [b]] = f[\pi_{\mathrm{ker}h}(a) \Rightarrow_{\mathrm{ker}h} \pi_{\mathrm{ker}h}(b)] = f[I(\pi_{\mathrm{ker}h}[a \Rightarrow_1 b])] =$$
$$f[\pi_{\mathrm{ker}h}[a \Rightarrow_1 b]] = h[a \Rightarrow_1 b] = I(h[a \Rightarrow_1 b]) = h(a) \Rightarrow_2 h(b) =$$
$$f(\pi_{\mathrm{ker}h}(a)) \Rightarrow_1 f(\pi_{\mathrm{ker}h}(b)) = f([a]) \Rightarrow_2 f([b]).$$

□

Let $\mathbf{A} \in \mathsf{DLatGi}$. We denote the family of the congruences of A by $\mathrm{Con}A$, and the family of the gi-congruences of $\mathbf{A}$ by $\mathrm{gCon}\mathbf{A}$.

4. REPRESENTATION FOR DISTRIBUTIVE LATTICES WITH A GENERALIZED IMPLICATION

Let $\mathbf{A}$ be a gi-lattice. We define the relation $R_{\mathbf{A}}$ on $\mathrm{Fi}(A)$ by:

$$(F, G) \in R_A \text{ iff } (\forall a, b \in A)((a \Rightarrow b) \cap F \neq \emptyset \ \& \ a \in G \Longrightarrow b \in G).$$

Let F be a filter of A. We define the operator $D_F : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$D_F(X) = \left\{ b \in A : \exists Y \subseteq X \text{ finite such that } (\bigwedge Y \Rightarrow b) \cap F \neq \emptyset \right\},$$

where $\bigwedge Y$ is the infimum of Y, so if Y is empty, $\bigwedge Y = 1$. Some useful properties of the operator $D_F(X)$ are listed in the proposition below.

Proposition 11. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and let* F *be a filter of* A. *Then*

(1) D_F *is a finitary closure operator.*

(2) *$D_F(X)$ is a filter of A, for every $X \subseteq A$.*
(3) *$(F, D_F(X)) \in R_A$, for every $X \subseteq A$.*
(4) *$D_F(X)$ is the least filter of A containing X such that $(F, D_F(X)) \in R_A$.*
(5) *If $(F, Q) \in R_A$ and Q is a prime filter, then there exists a prime filter P of A such that $F \subseteq P$ and $(P, Q) \in R_A$.*
(6) *For every filter G of A, $(F, G) \in R_A$ iff $D_F(G) = G$.*

Proof. 1. Let $a \in X \subseteq A$. Since $a \Rightarrow a = A$, then $a \in D_F(X)$. It is clear that if $X \subseteq Y \subseteq A$, $D_F(X) \subseteq D_F(Y)$. Now, let $a \in D_F(D_F(X))$. Thus, there are $c_0, \dots, c_n \in D_F(X)$ such that $(\bigwedge_{i \le n} c_i \Rightarrow a) \cap F \neq \emptyset$ or $(1 \Rightarrow a) \cap F \neq \emptyset$. Then, for each $i \le n$, there are $d^i_0, \dots, d^i_{k_i} \in X$ such that $(d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i) \cap F \neq \emptyset$ or $(1 \Rightarrow c_i) \cap F \neq \emptyset$. Since for each $i \le n$, $\bigwedge_{i \le n} d^i_0 \wedge \dots \wedge d^i_{k_i} \le d^i_0 \wedge \dots \wedge d^i_{k_i}$, then from 1 of Proposition 2 it follows that for each $i \le n$

$$(d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i) \subseteq (\bigwedge_{i \le n} d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i).$$

Using the facts that F is a filter, that for each $i \le n$ $(d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i)$ is an ideal, and that $(d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i) \cap F \neq \emptyset$, we conclude that

$$\bigcap_{i \le n} (\bigwedge_{i \le n} d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_i) \cap F \neq \emptyset.$$

Applying condition G1 of Definition 1, we obtain

$$(\bigwedge_{i \le n} d^i_0 \wedge \dots \wedge d^i_{k_i} \Rightarrow c_0 \wedge \dots \wedge c_n) \cap F \neq \emptyset.$$

Now from the the fact that $(\bigwedge_{i=1}^{n} c_i \Rightarrow a) \cap F \neq \emptyset$, using condition G3 of Definition 1, it follows that $a \in D_F(X)$. Finally, from the definition of D_F it follows that if $a \in D_F(X)$, then there is a finite $Y \subseteq X$ such that $a \in D_F(Y)$; hence D_F is finitary.

2. Let $X \subseteq A$. It is clear that $1 \in D_F(X)$ because for any finite subset $Y \subseteq X$, $\bigwedge Y \Rightarrow 1 = A$. Let $a, b \in D_F(X)$. Thus, there are finite subsets Y_1, Y_2 of X such that $(\bigwedge Y_1 \Rightarrow a) \cap F \neq \emptyset$ and $(\bigwedge Y_2 \Rightarrow b) \cap F \neq \emptyset$. Let $Y = Y_1 \cup Y_2$. As $\bigwedge Y \le \bigwedge Y_1$ and $\bigwedge Y \le \bigwedge Y_2$, then, applying (1) of Proposition 2, we have $(\bigwedge Y_1 \Rightarrow a) \subseteq (\bigwedge Y \Rightarrow a)$ and $(\bigwedge Y_2 \Rightarrow b) \subseteq (\bigwedge Y \Rightarrow b)$. Thus

$$(\bigwedge Y_1 \Rightarrow a) \cap (\bigwedge Y_2 \Rightarrow b) \cap F \subseteq (\bigwedge Y \Rightarrow a) \cap (\bigwedge Y \Rightarrow b) \cap F.$$

As we did in the proof of (1), we can prove that $(\bigwedge Y \Rightarrow a) \cap (\bigwedge Y \Rightarrow b) \cap F \neq \emptyset$. Applying condition G1 of Definition 1 it follows that $a \wedge b \in D_F(X)$. Now assume that $a \le b$ and $a \in D_F(X)$. Thus, there exists a finite $Y \subseteq X$ such that $(\bigwedge Y \Rightarrow a) \cap F \neq \emptyset$. Hence $(\bigwedge Y \Rightarrow a \wedge b) \cap F \neq \emptyset$, and by G1 we obtain $(\bigwedge Y \Rightarrow a) \cap (\bigwedge Y \Rightarrow b) \cap F \neq \emptyset$. Therefore $(\bigwedge Y \Rightarrow b) \cap F \neq \emptyset$.

3. Let $a, b \in A$ be such that $(a \Rightarrow b) \cap F \neq \emptyset$ and $a \in D_F(X)$. Consequently, there is a finite $Y \subseteq X$ such that $(\bigwedge Y \Rightarrow a) \cap F \neq \emptyset$. As in the proofs of (1) and (2), it can be shown that $(\bigwedge Y \Rightarrow a) \cap (a \Rightarrow b) \cap F \neq \emptyset$. Using condition G3 of Definition 1 it follows that $(\bigwedge Y \Rightarrow b) \cap F \neq \emptyset$. Thus, $b \in D_F(X)$.

4. Let $(F, G) \in R_A$, $X \subseteq G$ and $a \in D_F(X)$. Let $Y \subseteq X$ be a finite set such that $(\bigwedge Y \Rightarrow a) \cap F \neq \emptyset$. Since G is filter, $\bigwedge Y \in G$. Then from the hypothesis it follows that $a \in G$.

5. Assume that Q is a prime filter and $(F, Q) \in R_A$. Consider the family

$$\mathcal{F} = \{H \in \mathrm{Fi}(A) : F \subseteq H \text{ and } (H, Q) \in R_A\}.$$

Since $(F,Q) \in R_A$, $\mathcal{F}$ is non-empty. Notice that every chain of elements of $\mathcal{F}$, ordered by inclusion, has a supremum. Thus, by Zorn's lemma there exists a maximal element $P \in \mathcal{F}$. We show that P is a prime filter. Let $a \vee b \in P$ and suppose that $a, b \notin P$. Consider the filters $F_a = F(P \cup \{a\})$ and $F_b = F(P \cup \{b\})$. Since P is maximal on $\mathcal{F}$, $F_a, F_b \notin \mathcal{F}$. Hence, $(F_a, Q) \notin R_A$ and $(F_b, Q) \notin R_A$. Thus, there are $x_1, y_1 \in A$ such that $(x_1 \Rightarrow y_1) \cap F_a \neq \emptyset$, $x_1 \in Q$ and $y_1 \notin Q$. Moreover, there are $x_2, y_2 \in A$ such that $(x_2 \Rightarrow y_2) \cap F_b \neq \emptyset$, $x_2 \in Q$ and $y_2 \notin Q$. Since $(x_1 \Rightarrow y_1) \cap F_a \neq \emptyset$ and $(x_2 \Rightarrow y_2) \cap F_b \neq \emptyset$, there are $z_1, z_2 \in A$ with $z_1 \in (x_1 \Rightarrow y_1) \cap F_a$ and $z_2 \in (x_2 \Rightarrow y_2) \cap F_b$. Therefore, there exist $p_1, p_2 \in P$ such that $p_1 \wedge a \leq z_1$ and $p_2 \wedge b \leq z_2$. Then $p_1 \wedge a \in (x_1 \Rightarrow y_1)$ and $p_2 \wedge b \in (x_2 \Rightarrow y_2)$, because $x_1 \Rightarrow y_1$ and $x_2 \Rightarrow y_2$ are ideals. We take $p := p_1 \wedge p_2 \in P$. It is clear that $p \wedge a \in (x_1 \Rightarrow y_1)$ and $p \wedge b \in (x_2 \Rightarrow y_2)$. Now, we consider $x := x_1 \wedge x_2 \in Q$. Since $x \leq x_1$, we have that $x_1 \Rightarrow y_1 \subseteq x \Rightarrow y_1$. Analogously, $x_2 \Rightarrow y_2 \subseteq x \Rightarrow y_2$. Since $y_1, y_2 \notin Q$ and Q is a prime filter, we have $y := y_1 \vee y_2 \notin Q$. Since $y_1 \leq y$, then $(x \Rightarrow y_1) \subseteq (x \Rightarrow y)$. Similarly $(x \Rightarrow y_2) \subseteq (x \Rightarrow y)$. Thus, $p \wedge a \in (x_1 \Rightarrow y_1) \subseteq (x \Rightarrow y_1) \subseteq (x \Rightarrow y)$. In an analogous way we have $p \wedge b \in (x \Rightarrow y)$. Therefore $(p \wedge a) \vee (p \wedge b) = p \wedge (a \vee b) \in (x \Rightarrow y)$. Since $p \wedge (a \vee b) \in P$, we have $(x \Rightarrow y) \cap P \neq \emptyset$. Now, since $x \in Q$ and $(P,Q) \in R_A$, we obtain that $y \in Q$, which is a contradiction.

6. It is straightforward using the definitions. ☐

Let $\mathbf{A} \in \mathsf{DLatGi}$ and $X, Y \subseteq A$. We set

$$X \vee Y := \{a \vee b : a \in X \text{ and } b \in Y\}.$$

Proposition 12. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and* $X, Y \subseteq A$. *Then*

$$D_F(X) \cap D_F(Y) = D_F(X \vee Y).$$

Proof. Suppose $a \in D_F(X) \cap D_F(Y)$. Thus, there are $a_1, \ldots, a_n \in X$ and $b_1, \ldots, b_m \in Y$ such that

$$(\bigwedge_{i \leq n} a_i \Rightarrow a) \cap F \neq \emptyset \text{ and } (\bigwedge_{j \leq m} b_j \Rightarrow a) \cap F \neq \emptyset.$$

It is easy to see that the intersection of the two ideals with F is non-empty. Using condition G2 of Definition 1,

$$(((a_1 \wedge \ldots \wedge a_n) \vee (b_1 \wedge \ldots \wedge b_m)) \Rightarrow a) \cap F \neq \emptyset.$$

Thus, $a \in D_F(X \vee Y)$. To prove the other inclusion consider $a \in D_F(X \vee Y)$. Then, there exist $a_1, \ldots, a_n \in X$ and $b_1, \ldots, b_n \in Y$ such that

$$(((a_1 \vee b_1) \wedge \ldots \wedge (a_n \vee b_n)) \Rightarrow a) \cap F \neq \emptyset.$$

Using the distributivity laws and applying condition G2 of Definition 1,

$$((a_1 \wedge \ldots \wedge a_n) \Rightarrow a) \cap ((b_1 \wedge \ldots \wedge b_n) \Rightarrow a) \cap F \neq \emptyset.$$

So, $a \in D_F(X)$ and $a \in D_F(Y)$. ☐

Lemma 13 (Existence of prime filters)**.** *Let* $\mathbf{A} \in \mathsf{DLatGi}$, $X \subseteq A$, *and let* F *be a filter and* I *an ideal of* A. *If* $D_F(X) \cap I = \emptyset$, *then there exists a prime filter* P *such that*

(1) $D_F(X) \subseteq P$,
(2) $(F, P) \in R_A$,
(3) $P \cap I = \emptyset$.

Proof. We consider the family

$$\mathcal{F} := \{G \in \mathrm{Fi}(A) : (F, G) \in R_A,\ X \subseteq G \text{ and } G \cap I = \emptyset\}.$$

Since $D_F(X) \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. By Zorn's lemma there is a maximal element in $\mathcal{F}$, because every chain of elements of $\mathcal{F}$, ordered by inclusion, has a supremum. Let

P be such an element. Clearly P is proper and $D_F(P) = P$. We prove that P is prime. We suppose the opposite. Thus, there are $a, b \in A$ such that $a \vee b \in P$ and $a, b \notin P$. Let us consider the sets $P \cup \{a\}$ and $P \cup \{b\}$. As P is a maximal element of $\mathcal{F}$ we have $D_F(P \cup \{a\}) \cap I \neq \emptyset$ and $D_F(P \cup \{b\}) \cap I \neq \emptyset$. In addition, as I is an ideal, we conclude that $D_F(P \cup \{a\}) \cap D_F(P \cup \{b\}) \cap I \neq \emptyset$. Using Proposition 12 we obtain $D_F(P \cup \{a \vee b\}) \cap I \neq \emptyset$. Thus, $D_F(P) \cap I \neq \emptyset$ because $a \vee b \in P$. Therefore $P \cap I \neq \emptyset$, which is a contradiction. So, $P \in X(A)$ and moreover $D_F(X) \subseteq P$, $(F, P) \in R_A$ and $P \cap I = \emptyset$. □

Remark 14. The above lemma is a generalization of the prime filter theorem for bounded distributive lattices. Indeed, given a bounded distributive lattice A, let us define $\Rightarrow: A \times A \to \mathrm{Id}(A)$ as in Example 3. If $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$ are such that $F \cap I = \emptyset$, then $D_{\{1\}}(F) \cap I = \emptyset$, because otherwise there is $x \in I$ such that $(a \Rightarrow x) \cap \{1\} \neq \emptyset$, for some $a \in F$. Since $a \Rightarrow x$ is an ideal that contains 1, we have $a \Rightarrow x = A$. Thus $a \leq x$, and so $a \in I$. Therefore, $F \cap I \neq \emptyset$ which is a contradiction. Now, since $\{1\}$ is a filter, I is an ideal and as $D_{\{1\}}(F) \cap I = \emptyset$, Lemma 13 implies that there exists $P \in X(A)$ such that $D_{\{1\}}(F) \subseteq P$ and $P \cap I = \emptyset$. Since $F \subseteq D_{\{1\}}(F)$ we have a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.

Lemma 15. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*,* $a, b \in A$ *and* $P \in X(A)$*. Then* $(a \Rightarrow b) \cap P = \emptyset$ *if and only if there exits* $Q \in X(A)$ *such that* $(P, Q) \in R_A$*,* $a \in Q$ *and* $b \notin Q$*.*

Proof. If $(a \Rightarrow b) \cap P = \emptyset$, then $D_P(a) \cap {\downarrow} b = \emptyset$. So, there exists $Q \in X(A)$ such that $(P, Q) \in R_A$, $a \in Q$ and $b \notin Q$. The other implication follows immediately from the definition of R_A. □

Our aim now is to prove a representation theorem for gi-lattices. Let $\mathbf{A} = \langle A, \Rightarrow \rangle \in \mathsf{DLatGi}$. Since A is a distributive lattice, let $X(A)$ be its Priestley space. Recall that the Stone map $\beta : A \to \mathcal{P}_u(X(A))$ is an isomorphism between A and the bounded distributive lattice $\beta[A]$. We will show that $\mathbf{A}$ is isomorphic to the gi-lattice $\langle \beta[A], \Rightarrow_{R_A}^{\beta[A]} \rangle$, where $\Rightarrow_{R_A}^{\beta[A]}$ is defined as in Example 2. For simplicity we abbreviate $\Rightarrow_{R_A}^{\beta[A]}$ by $\Rightarrow_{R_A}$. First we need a lemma.

Lemma 16. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. For every* $a, b \in A$

$$\beta(a) \to_{R_A} \beta(b) = \bigcup \beta[a \Rightarrow b] = \{P \in X(A) : P \cap a \Rightarrow b \neq \emptyset\}.$$

Proof. Let $P \in \beta(a) \to_{R_A} \beta(b)$ and suppose that $P \notin \beta[a \Rightarrow b]$, that is $P \notin \bigcup_{c \in (a \Rightarrow b)} \beta(c)$. It follows that $P \cap (a \Rightarrow b) = \emptyset$. Using Lemma 15 there is $Q \in X(\mathbf{A})$ such that $(P, Q) \in R_A$, $Q \in \beta(a)$ and $Q \notin \beta(b)$, which is a contradiction because $P \in \beta(a) \to_{R_A} \beta(b)$. In order to prove the other inclusion let $P \in \bigcup_{c \in (a \Rightarrow b)} \beta(c)$. Thus, $P \cap (a \Rightarrow b) \neq \emptyset$. In order to prove that $P \in \beta(a) \to_{R_A} \beta(b)$ suppose that $Q \in R_A(P) \cap \beta(a)$. Then, $(P, Q) \in R_A$ and $a \in Q$. Hence, since $P \cap (a \Rightarrow b) \neq \emptyset$, from the definition of R_A follows that $b \in Q$ and so $Q \in \beta(b)$. □

Recall that if $X(A)$ is the Priestley space of a bounded distributive lattice A, then the map φ from the set of ideals of A to the set of open up-sets of $X(A)$ defined by

$$\varphi(I) = \{P \in X(A) : P \cap I \neq \emptyset\},$$

for every $I \in \mathrm{Id(A)}$, is a lattice isomorphism. Thus, Lemma 16 can be restated as follows.

Corollary 17. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. For every* $a, b \in A$*.*

$$\beta(a) \to_{R_A} \beta(b) = \varphi(a \Rightarrow b).$$

Theorem 18 (Representation theorem)**.** *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. Then* β *is a gi-isomorphism between* $\mathbf{A}$ *and* $\langle \beta[A], \Rightarrow_{R_A} \rangle$*.*

Proof. From Priestley duality we know that β is a bounded lattice isomorphism between A and $\beta[A]$. It remains to show that $I(\beta\,[a \Rightarrow b]) = \beta(a) \Rightarrow_{R_A} \beta(b)$. Since β is an isomorphism and $a \Rightarrow b$ is an ideal of $\mathbf{A}$, $\beta\,[a \Rightarrow b]$ is an ideal of $\beta[A]$. Therefore, $I(\beta\,[a \Rightarrow b]) = \beta\,[a \Rightarrow b]$. Let us prove that $\beta\,[a \Rightarrow b] = \beta(a) \Rightarrow_{R_A} \beta(b)$. Assume that $c \in a \Rightarrow b$. We show that $\beta(c) \in \beta(a) \Rightarrow_{R_A} \beta(b)$. We have to prove that for every $P \in \beta(c)$, $R_A(P) \cap \beta(a) \subseteq \beta(b)$. Let $P \in \beta(c)$ and assume that $Q \in R_A(P) \cap \beta(a)$. Then since $P \cap (a \Rightarrow b) \neq \emptyset$ it follows that $b \in Q$ and so $Q \in \beta(b)$. Thus we have proved that $\beta\,[a \Rightarrow b] \subseteq \beta(a) \Rightarrow_{R_A} \beta(b)$. To prove the other inclusion let $\beta(c) \in \beta(a) \Rightarrow_{R_A} \beta(b)$. Hence, $\beta(c) \subseteq \beta(a) \rightarrow_{R_A} \beta(b)$. Consequently, by Lemma 16 we have $\beta(c) \subseteq \bigcup \beta[a \Rightarrow b]$. Suppose now that $c \notin a \Rightarrow b$. Then by the prime filter lemma there is $P \in X(A)$ such that $c \in P$ and $P \cap (a \Rightarrow b) = \emptyset$. Therefore, $P \in \beta(c)$ and hence, $P \in \bigcup \beta[a \Rightarrow b]$. This implies that $P \cap (a \Rightarrow b) \neq \emptyset$. A contradiction. □

5. A TOPOLOGICAL DUALITY FOR DISTRIBUTIVE LATTICES WITH A GENERALIZED IMPLICATION

Our next aim is to develop a topological duality for distributive lattices with a generalized implication. The dual of a gi-lattice $\mathbf{A} = \langle A, \Rightarrow \rangle$ will be the Priestley space $X(A)$ of A together with an additional binary relation that encodes the behaviour of $\Rightarrow$. We will obtain a category dually equivalent to the category **DLatGi**. In order to justify the definition of the dual of a gi-lattice we first establish the following results.

Proposition 19. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. Then for every* $P \in X(A)$*,* $R_{\mathbf{A}}(P)$ *is a closet set of the Priestley space* $X(A)$*.*

Proof. Let $P \in X(P)$ and assume that $Q \notin R_A(P)$ with $Q \in X(A)$. By definition of R_A, there are $a, b \in A$ such that $a \Rightarrow b \cap P \neq \emptyset$, $a \in Q$ and $b \notin Q$. Since $\beta(a) \rightarrow_{R_A} \beta(b) = \bigcup \beta[a \Rightarrow b]$, we get that $R_A(P) \cap \beta(a) \subseteq \beta(b)$. Thus, $R_A(P) \subseteq \beta^c(a) \cup \beta(b)$, and $Q \notin \beta^c(a) \cup \beta(b)$. As $\beta^c(a) \cup \beta(b)$ is a closed subset of $X(\mathbf{A})$, we deduce that Q is not in the closure of $R_A(P)$. It follows that $R_A(P)$ is a closed set of $X(\mathbf{A})$. □

Proposition 20. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. Then for every* $a, b \in A$*,* $\beta(a) \rightarrow_{R_A} \beta(b)$ *is an open up-set of the Priestley space* $X(A)$*.*

Proof. It follows immediately from Corollary 17. □

The two propositions justify the following definition.

Definition 21. A *gi-space* is a structure $X = \langle X, \leq, \tau_X, R \rangle$ where $\langle X, \leq, \tau_X \rangle$ is a Priestley space and R is a binary relation on X such that:

(1) $R(x)$ is a closed set, for every $x \in X$.
(2) For all $U, V \in CU(X)$, $U \rightarrow_R V$ is an open up-set of X.

From Priestley duality and Propositions 19 and 20 we obtain:

Proposition 22. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. Then* $X(\mathbf{A}) = \langle X(A), \subseteq, \tau_X(A), R_A \rangle$ *is a gi-space.*

Now we define the gi-lattice associated with a gi-space. Let $X = \langle X, \leq, \tau_X, R \rangle$ be a gi-space and let $U, V \in CU(X)$. We define the set

$$U \Rightarrow_R V := \{W \in CU(X) : W \subseteq (U \rightarrow_R V)\},$$

which is an ideal of $CU(X)$. This leads us to the following result. For brevity we will denote $CU(X)$ by D_X.

Lemma 23. *Let* $X = \langle X, \leq, \tau_X, R\rangle$ *be a gi-space. Then* $\langle D_X, \Rightarrow_R\rangle$ *is a gi-lattice.*

Proof. We prove that conditions G1-G4 of Definition 1 hold. To prove that G1 holds, let $S \in (U \Rightarrow_R V) \cap (U \Rightarrow_R T)$. Thus, $R(x) \cap U \subseteq V$ and $R(x) \cap U \subseteq T$, for each $x \in S$. Therefore $R(x) \cap U \subseteq V \cap T$. Hence, $S \in [U \Rightarrow_R (V \cap T)]$. The other inclusion is shown without any difficulty.

In order to prove condition G2, let $S \in (U \Rightarrow_R V) \cap (T \Rightarrow_R V)$. Thus, for each $x \in S$ we have $R(x) \cap U \subseteq V$ and $R(x) \cap T \subseteq V$. Hence, $R(x) \cap (U \cup T) \subseteq V$. This implies that $S \in (U \cup T) \Rightarrow_R V$. The other inclusion is immediate, using distributivity.

Now we prove G3. Let $S \in (U \Rightarrow_R V) \cap (V \Rightarrow_R T)$. Therefore, for each $x \in S$ we have that $R(x) \cap U \subseteq V$ and $R(x) \cap V \subseteq T$. Thus, $R(x) \cap U \subseteq T$. Hence $S \in U \Rightarrow_R T$.

Finally we prove that G4 holds. For each $x \in S$, we have $R(x) \cap U \subseteq U$. Therefore, $(U \rightarrow_R U) = X$. So, $U \Rightarrow_R U = D$. ☐

If X is a gi-space, we denote by X^* the gi-lattice $\langle D_X, \Rightarrow_R\rangle$.

In order to obtain a categorical duality between gi-lattices and gi-spaces, we need to introduce a notion of morphism between gi-spaces.

Definition 24. Let $X_1 = \langle X_1, \leq_1, \tau_{X_1}, R_1, \rangle$ and $X_2 = \langle X_2, \leq_2, \tau_{X_2}, R_2\rangle$ be gi-spaces. We say that a function $f : X_1 \rightarrow X_2$ is a *gi-morphism* from X_1 to X_2 if it is a continuous and monotone function such that:

(1) if $(x, y) \in R_1$ then $(f(x), f(y)) \in R_2$,
(2) if $(f(x), y) \in R_2$, then there exists $z \in X_1$ such that $(x, z) \in R_1$ and $f(z) = y$.

We say that f is a *gi-isomorphism* if it is a gi-morphism which is also an order-isomorphism and a homeomorphism.

Remark 25. It is easy to check that the composition of gi-morphisms is a gi-morphism, and if $X = \langle X, \leq, \tau_X, R\rangle$ is a gi-space, then Id_X (the identity map) is a gi-morphism. Therefore the class of all gi-spaces taken as objects and all the gi-morphisms between them form a category that we denote by **GiS**.

Recall that given a Priestely space X, the map $\varepsilon_X : X \rightarrow X(CU(X))$ defined by

$$\varepsilon_X(x) = \{U \in CU(X) : x \in U\},$$

for every $x \in X$, is a homeomorphism and an order-isomorphism.

Lemma 26. *Let* $X = \langle X, \leq, \tau_X, R\rangle$ *be a gi-space. Then*

$$(x, y) \in R \quad \textit{iff} \quad (\varepsilon_X(x), \varepsilon_X(y)) \in R_{X^*}.$$

Proof. Suppose that $(x, y) \in R$. Let $U, V \in D_X$ be such that $U \in \varepsilon_X(y)$ and $(U \Rightarrow_R V) \cap \varepsilon_X(x) \neq \emptyset$. Thus, there exists $Z \in D_X$ such that $x \in Z$ and $Z \subseteq \{z : R(z) \cap U \subseteq V\}$. Therefore $V \in \varepsilon_X(y)$. Now, suppose that $(\varepsilon_X(x), \varepsilon_X(y)) \in R_{X^*}$ and $y \notin R(x)$. Since $R(x)$ is a closed set, there are $U, V \in D_X$ such that $R(x) \cap (U - V) = \emptyset$ and $y \in (U - V)$. Thus, $(U \Rightarrow_R V) \cap \varepsilon_X(x) \neq \emptyset$, $U \in \varepsilon_X(y)$ and $V \notin \varepsilon_X(y)$, which contradicts the assumption that $(\varepsilon_X(x), \varepsilon_X(y)) \in R_{X^*}$. ☐

Theorem 27. *Let* $X = \langle X, \leq, \tau_X, R\rangle$ *be a gi-space. Then the map* $\varepsilon_X : X \rightarrow X(X^*)$ *is a gi-isomorphism.*

Proof. By Priestley duality we know that ε_X is an order-isomorphism, and a homeomorphism. By Lemma 26 we have $(x, y) \in R$ *iff* $(\varepsilon_X(x), \varepsilon_X(y)) \in R_{X^*}$. It follows that ε_X is a gi-isomorphism. ☐

Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathsf{DLatGi}$ and let h be a bounded lattice homomorphism from A_1 to A_2. Recall that the map $h_* : X(A_2) \to X(A_1)$ is defined by $h_*(P) = h^{-1}(P)$, for each $P \in X(A_2)$.

Theorem 28. *Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathsf{DLatGi}$. A lattice homomorphism $h : A_1 \to A_2$ is a gi-homomorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$ if and only if the map h_* is a gi-morphism from $X(\mathbf{A}_2)$ to $X(\mathbf{A}_1)$.*

Proof. $\Rightarrow$) From Priestley duality for distributive lattices we know that h_* is continuous and monotone. To prove condition (1) of the definition of g-morphism, let $P, Q \in X(A_2)$ be such that $(P, Q) \in R_{A_2}$. To show that $(h_*(P), h_*(Q) \in R_{A_1}$, let $a, b \in A_1$ be such that $(a \Rightarrow_1 b) \cap h^{-1}(P) \neq \emptyset$ and $a \in h^{-1}(Q)$. Using the fact that h is a gi-homomorphism, we obtain that $(h(a) \Rightarrow_2 h(b)) \cap P \neq \emptyset$. Moreover, since $h(a) \in Q$ and $(P, Q) \in R_{A_2}$, it follows that $h(b) \in Q$, and so $b \in h^{-1}(Q)$. To prove condition (2), suppose that $(h^{-1}(P), Q) \in R_{A_1}$. We consider the filter $D_P(h[Q])$. Since h is a lattice homomorphism, $h[Q]$ is closed under $\wedge$; hence

$$D_P(h[Q]) = \{y : \exists q \in Q\ (h(q) \Rightarrow_2 y) \cap P \neq \emptyset\}.$$

We show that $D_P(h[Q]) \cap I(h(Q^c)) = \emptyset$. Suppose the opposite. Then let $x \in D_P(h[Q])$, $q \in Q$ and $a \notin Q$ be such that $(h(q) \Rightarrow_2 x) \cap P \neq \emptyset$ and $x \leq h(a)$. Thus,

$$(h(q) \Rightarrow_2 x) \cap P \subseteq (h(q) \Rightarrow_2 h(a)) \cap P = I(h[q \Rightarrow_1 a] \cap P) \neq \emptyset.$$

This implies that $(q \Rightarrow_1 a) \cap h^{-1}(P) \neq \emptyset$. From the assumption that $(h^{-1}(P), Q) \in R_{A_1}$ and $q \in Q$ follows that $a \in Q$, a contradiction. Therefore, applying Lemma 13 there is $Z \in X(A_2)$ such that $(P, Z) \in R_{A_2}$ and $h^{-1}(Z) = Q$.

$\Leftarrow$) Since h is a lattice homomorphism, we only need to prove that for every $a, b \in A_1$, $I(h[a \Rightarrow_1 b]) = h(a) \Rightarrow_2 h(b)$. This is equivalent to showing that $\beta[I(h[a \Rightarrow_1 b])] = \beta[h(a) \Rightarrow_2 h(b)]$. Let $P \in X(A_2)$ be such that $P \cap I(h[a \Rightarrow_1 b]) \neq \emptyset$. Thus, there are $x \in P$ and $z \in (a \Rightarrow_1 b)$ such that $x \leq h(z)$. Since P is a filter, $z \in h^{-1}(P)$. Therefore, $h^{-1}(P) \cap (a \Rightarrow_1 b) \neq \emptyset$. Now we prove that $P \cap (h(a) \Rightarrow_2 h(b)) \neq \emptyset$. If we assume the opposite, then by Lemma 15 there exists $Q \in X(\mathbf{A}_2)$ such that $(P, Q) \in R_{A_2}$, $a \in h^{-1}(Q)$ and $b \notin h^{-1}(Q)$. Using the fact that h_* is a gi-morphism, we have

$$(h^{-1}(P), h^{-1}(Q)) \in R_{A_1}.$$

Therefore, since $h^{-1}(P) \cap (a \Rightarrow_1 b) \neq \emptyset$ and $a \in h^{-1}(Q)$, we obtain $b \in h^{-1}(Q)$, which is a contradiction. In order to prove the other inclusion, let $P \in X(A_2)$ be such that $P \cap (h(a) \Rightarrow_2 h(b)) \neq \emptyset$. It is enough to show that $P \cap h[a \Rightarrow_1 b] \neq \emptyset$. Let us assume the opposite. Thus $h^{-1}(P) \cap (a \Rightarrow_1 b) = \emptyset$. By Lemma 15, there is $Q \in X(A_1)$ such that $(h^{-1}(P), Q) \in R_{A_1}$, $a \in Q$ and $b \notin Q$. Using that h_* is a gi-morphism, it follows that there exists $Z \in X(A_2)$ such that $(P, Z) \in R_{A_2}$ and $h^{-1}(Z) = Q$. Moreover, since $P \cap (h(a) \Rightarrow_2 h(b)) \neq \emptyset$ and $a \in Q$, we obtain that $h(b) \in Z$. So, $b \in Q$, which is a contradiction. □

Lemma 29. *Let $\mathbf{X}_1 = \langle X_1, \leq_1, \tau_{X_1}, R_1 \rangle$ and $\mathbf{X}_2 = \langle X_2, \leq_2, \tau_{X_2}, R_2 \rangle$ be gi-spaces and $f : X_1 \to X_2$ a map such that for every $U \in D_{X_2}$, $f^{-1}(U) \in D_{X_1}$. Then the following conditions are equivalent for every $U, V \in D_{X_2}$,*

1. $I(f^{-1}(U \Rightarrow_{R_2} V)) = f^{-1}(U) \Rightarrow_{R_1} f^{-1}(V)$.
2. $f^{-1}(U) \to_{R_1} f^{-1}(V) = f^{-1}(U \to_{R_2} V)$.

Proof. (1) implies (2). Let $x \in f^{-1}(U) \to_{R_1} f^{-1}(V)$. Since by assumption $f^{-1}(U), f^{-1}(V) \in D_{X_1}$, $f^{-1}(U) \to_{R_1} f^{-1}(V)$ is an open up-set of X_1. Therefore it is a union of elements of D_{X_1}. Let $W \in D_{X_1}$ be such that $x \in W \subseteq f^{-1}(U) \to_{R_1} f^{-1}(V)$. Then, $W \in f^{-1}(U) \Rightarrow_{R_1} f^{-1}(V) = I(f^{-1}(U \Rightarrow_{R_2} V))$. So, there is $Z \in D_{X_1}$ such that $W \subseteq Z$ and $Z \in f^{-1}(U \Rightarrow_{R_2} V)$. Therefore, there exists $Y \in D_{X_2}$ such that

$Y \subseteq U \to_{R_2} V$ and $Z = f^{-1}(Y)$. It follows that $f(x) \in Y$ and since $Y \subseteq U \to_{R_2} V$, $x \in f^{-1}(U \to_{R_2} V)$. One proves the other inclusion in a similar fashion.

(2) implies (1). Let $W \in I(f^{-1}(U \Rightarrow_{R_2} V))$. Thus, there exists $Z \in D_{X_1}$ such that $W \subseteq Z$ and $Z \in f^{-1}(U \Rightarrow_{R_2} V)$. Therefore, there is $Y \in D_{X_2}$ such that $Y \subseteq U \to_{R_2} V$ and $Z = f^{-1}(Y)$. Then, $W \subseteq f^{-1}(Y)$. Let $x \in W$. Hence $f(x) \in Y \subseteq U \to_{R_2} V$. Therefore, $W \subseteq f^{-1}(U \to_{R_2} V)$ and $W \subseteq f^{-1}(U) \to_{R_1} f^{-1}(V)$. So, $W \in f^{-1}(U) \Rightarrow_{R_1} f^{-1}(V)$. The other inclusion is obtained in a similar way. □

Let $\mathbf{X}_1 = \langle X_1, \leq_1, \tau_{X_1}, R_1\rangle$ and $\mathbf{X}_2 = \langle X_2, \leq_2, \tau_{X_2}, R_2\rangle$ be gi-spaces and f a map from X_1 to X_2. Recall that the function $f^* : D_{X_2} \to D_{X_1}$ is defined by $f^*(U) = f^{-1}(U)$, for every $U \in D_{X_2}$.

Theorem 30. *Let* $\mathbf{X}_1 = \langle X_1, \leq_1, \tau_{X_1}, R_1\rangle$ *and* $\mathbf{X}_2 = \langle X_2, \leq_2, \tau_{X_2}, R_2\rangle$ *be gi-spaces. A map* $f : X_1 \to X_2$ *is a gi-morphism from* X_1 *to* X_2 *if and only if the function* f^* *is a gi-homomorphism from* X_2^* *to* X_1^*.

Proof. ⇒) According to Lemma 29, it suffices to prove that $f^{-1}(U) \to_{R_1} f^{-1}(V) = f^{-1}(U \to_{R_2} V)$, for every $U, V \in D_{X_2}$. Let $x \in f^{-1}(U) \to_{R_1} f^{-1}(V)$. Thus, $R_1(x) \cap f^{-1}(U) \subseteq f^{-1}(V)$. We prove that $R_2(f(x)) \cap U \subseteq V$. Let $y \in U$ be such that $(f(x), y) \in R_2$. Since f is a gi-morphism, there exists $z \in X_1$ with $(x, z) \in R_1$ and $y = f(z)$. Hence, $z \in R_1(x)$ and $z \in f^{-1}(U)$. Thus, $z \in f^{-1}(V)$ and so $y \in V$. To prove the other inclusion, let $x \in f^{-1}(U \to_{R_2} V)$. Therefore, $R_2(f(x)) \cap U \subseteq V$. Let $y \in R_1(x) \cap f^{-1}(U)$. Since f is a gi-morphism, we have $(f(x), f(y)) \in R_2$. Thus, $y \in f^{-1}(V)$.

⇐) We prove conditions (1) and (2) of the definition of gi-morphism. To prove (1) let $(x, y) \in R_1$ and suppose that $(f(x), f(y)) \notin R_2$. Since $R_2(f(x))$ is a closed set, there are $U, V \in D_{X_2}$ such that $R_2(f(x)) \cap (U - V) = \emptyset$ and $f(y) \in (U - V)$. In other words, $R_2(f(x)) \cap U \subseteq V$ and $y \in f^{-1}(U) \cap f^{-1}(V)^c$. Thus, $x \in f^{-1}(U \to_{R_2} V)$. Moreover, $x \notin (f^{-1}(U) \to_{R_1} f^{-1}(V))$, because $y \in R_1(x) \cap f^{-1}(U)$ and $y \notin f^{-1}(V)$. Therefore, $f^{-1}(U) \to_{R_1} f^{-1}(V) \neq f^{-1}(U \to_{R_2} V)$. Thus, applying Lemma 29, $I(f^{-1}(U \Rightarrow_2 V)) \neq f^{-1}(U) \Rightarrow_1 f^{-1}(V)$, which is impossible because f^* is by assumption a gi-homomorphism. Let us now prove condition (2). Assume that $(f(x), y) \in R_2$ and that for any $z \in R_1(x)$, $y \neq f(z)$. Thus, for every $z \in R_1(x)$, $f(z) \nleq_2 y$ or $y \nleq_2 f(z)$. Let $z \in R_1(x)$. If $f(z) \nleq_2 y$, let $V_z \in D_{X_2}$ such that $f(z) \in V_z$ and $y \notin V_z$, and if $y \nleq_2 f(z)$, let $U_z \in D_{X_2}$ such that $y \in U_z$ and $f(z) \notin U_z$. They exist because X_1 and X_2 are Priestley spaces. Now consider the sets

$$\mathcal{V} := \{V_z \in D_{X_2} : z \in R_1(x) \text{ and } f(z) \nleq_2 y\}$$

and

$$\mathcal{U} := \{U_z \in D_{X_2} : z \in R_1(x) \text{ and } y \nleq_2 f(z)\}.$$

Then

$$R_1(x) \subseteq \bigcup_{V_z \in \mathcal{V}} f^{-1}(V_z) \cup \bigcup_{U_z \in \mathcal{U}} (f^{-1}(U_z))^c. \tag{5.1}$$

Since $R_1(x)$ is a closed set and X_1 is compact, $R_1(x)$ is a compact set; thus (5.1) implies that there are $U, V \in D_{X_2}$ such that $R_1(x) \subseteq f^{-1}(V) \cup (f^{-1}(U))^c$. Therefore, $R_1(x) \cap f^{-1}(U) \subseteq f^{-1}(V)$. Hence $x \in f^{-1}(U) \to_{R_1} f^{-1}(V)$. Since f^* is a gi-homomorphism, $I(f^{-1}(U \Rightarrow_{R_2} V)) = f^{-1}(U) \Rightarrow_{R_1} f^{-1}(V)$. By Lemma 29, $f(x) \in (U \to_{R_2} V)$. Since $y \in R_2(f(x)) \cap U$, we have $y \in V$, which is a contradiction. □

The results of the section imply that the categories **DLatGi** and **GiS** are dually equivalent by the following contravariant functors $(.)_* : \mathbf{DLatGi} \to \mathbf{GiS}$ and $(.)^* : \mathbf{GiS} \to \mathbf{DLatGi}$. The functor $(.)_* : \mathbf{DLatGi} \to \mathbf{GiS}$ is defined as follows: for every $\mathbf{A} \in \mathsf{DLatGi}$, $(\mathbf{A})_* = X(\mathbf{A})$ and for every gi-homomorphism $h : \mathbf{A}_1 \to \mathbf{A}_2$,

$(h)_* = h_*$. The functor $(.)^* : \mathbf{GiS} \to \mathbf{DLatGi}$ is defined by: for every gi-space X, $(X)^* = X^*$, and for every g-morphism $f : X_1 \to X_2$, $(f)^* = f^*$.

The theorem below follows from the results given in this section.

Theorem 31. *The categories* **DLatGi** *and* **GiS** *are dually equivalent by the contravariant functors* $(.)_*$ *and* $(.)^*$.

6. An application of the duality

It is well known that the complete lattice of the congruences of a bounded ditributive lattice A is dually isomorphic to the complete lattice of the closed subsets of its dual Priestley space $X(A)$. The isomorphism is given by the map $\Theta(.)$ from the closed sets of $X(A)$ to the set Con$A$, defined as follows. For every closed set Y of $X(A)$

$$(a, b) \in \Theta(Y) \text{ iff } \beta(a) \cap Y = \beta(b) \cap Y$$

for every $a, b \in A$. We characterize the closed sets of $X(A)$ that correspond to gi-congruences.

If $\mathbf{A} \in \mathsf{DLatGi}$, recall that for every $a, b \in A$

$$\bigcup \beta[a \Rightarrow b] = \{P \in X(A) : P \cap (a \Rightarrow b) \neq \emptyset\}.$$

Lemma 32. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and let* Y *be a closed set of* $X(A)$*. Then for every* $(a, b), (c, d) \in \Theta(Y)$*, the following conditions are equivalent,*

(1) $\langle a \Rightarrow c, b \Rightarrow d\rangle \in \theta$,
(2) $\bigcup \beta[a \Rightarrow c] \cap Y = \bigcup \beta[b \Rightarrow d] \cap Y$.

Proof. Assume (1). First we prove that $\varphi(a \Rightarrow c) \cap Y \subseteq \varphi(b \Rightarrow d) \cap Y$. Suppose $P \in \bigcup \beta[a \Rightarrow c] \cap Y$. Let $d \in (a \Rightarrow c)$ be such that $P \in \beta(d)$. By Definition 7, there is $e \in (b \Rightarrow d)$ such that $(d, e) \in \Theta(Y)$. Thus, $\beta(d) \cap Y = \beta(e) \cap Y$. Since $P \in \beta(d) \cap Y$, $e \in P$, and therefore $P \in \beta(e) \subseteq \bigcup \beta[b \Rightarrow d]$. Hence, $P \in \bigcup \beta[b \Rightarrow d] \cap Y$. The other inclusion is proved similarly, using condition 2 on Definition 7.

Now, assume (2). Therefore

$$\bigcup_{d \in (a \Rightarrow c)} (\beta(d) \cap Y) = \bigcup_{e \in (b \Rightarrow d)} (\beta(e) \cap Y).$$

To prove condition 1 on Definition 7, let $d \in (a \Rightarrow c)$. Then,

$$\beta(d) \cap Y \subseteq \bigcup_{d \in (a \Rightarrow c)} (\beta(d) \cap Y) = \bigcup_{e \in (b \Rightarrow d)} (\beta(e) \cap Y) \subseteq \bigcup_{e \in (b \Rightarrow d)} \beta(e).$$

By compactness of the space $X(A)$ there are $e_1, \ldots, e_n \in (b \Rightarrow d)$ such that $\beta(d) \cap Y \subseteq \beta(e_1) \cup \ldots \cup \beta(e_n)$. Let $e = e_1 \vee \ldots \vee e_n$. Thus, $\beta(d) \cap Y \subseteq \beta(e) \cap Y$. Therefore $\beta(d) \cap Y = \beta(d) \cap Y \cap \beta(e) = \beta(d \wedge e) \cap Y$. Let $y = d \wedge e$. Consequently $(d, y) \in \theta(Y)$. Moreover, since $y \leq e$ and $e \in (b \Rightarrow d)$, we obtain that $y \in (b \Rightarrow d)$. Condition 2 on Definition 7 can be proved analogously. $\square$

Let X be a set and R a binary relation on X; a set $Y \subseteq X$ is *R-closed* if for every $x \in Y$, $R(x) \subseteq Y$.

Let $X = \langle X, \leq, \tau_X, R\rangle$. It is not difficult to see that the set of all closed sets of X which are R-closed forms a complete sublattice of the complete lattice of the closed sets of X, where the infimum of a family is its intersection.

Lemma 33. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and let* Y *be a closed set of* $X(A)$*. Then* $\Theta(Y)$ *is a gi-congruence of* $\mathbf{A}$ *if and only if* Y *is* R_A*-closed.*

Proof. Suppose that $\Theta(Y)$ is a gi-congruence. Let $P \in Y$, $Q \in R_A(P)$ and suppose that $Q \notin Y$. Since Y is a closed set, there are $a, b \in A$ such that $Y \subseteq \beta(a)^c \cup \beta(b)$ and $Q \notin \beta(a)^c \cup \beta(b)$. Then $\beta(a) \cap Y = \beta(a) \cap \beta(b) \cap Y$. Hence we have $(a, a \wedge b) \in \theta(Y)$, $a \in Q$ and $b \notin Q$. Since $\Theta(Y)$ is a congruence, $(b, b) \in \Theta(Y)$ and therefore, since by assumption $\Theta(Y)$ is a gi-congruence, we have $\langle a \Rightarrow b, (a \wedge b) \Rightarrow b\rangle \in \Theta(Y)$. The lemma above implies that

$$\bigcup \beta[a \Rightarrow b] \cap Y = \bigcup \beta[(a \wedge b) \Rightarrow b] \cap Y.$$

Now, since $a \wedge b \leq b$, $(a \wedge b) \Rightarrow b = A$. So, $\bigcup \beta[(a \wedge b) \Rightarrow b] \cap Y = Y$. Therefore, $Y \subseteq \bigcup \beta[a \Rightarrow b]$. Thus, $(a \Rightarrow b) \cap P \neq \emptyset$ because $P \in Y$. Moreover, since $(P, Q) \in R_A$ and $a \in Q$ it follows that $b \in Q$, which is a contradiction.

Suppose now that Y is a R_A-closed set. To show that $\theta(Y)$ is a gi-congruence, assume that $(a_1, b_1), (a_2, b_2) \in \theta(Y)$ and that $P \in \bigcup \beta[a_1 \Rightarrow a_2] \cap Y$. Thus, $P \cap (a_1 \Rightarrow a_2) \neq \emptyset$ and $P \in Y$. Suppose that $P \cap (b_1 \Rightarrow b_2) = \emptyset$. Then, by Lemma 15, there exists $Q \in X(A)$ such that $(P, Q) \in R_A$, $b_1 \in Q$ and $b_2 \notin Q$. Since Y is a R_A-closed set, it follows that $Q \in Y$. Therefore, $Q \in \beta(b_1) \cap Y$, and since $(a_1, a_1) \in \Theta(Y)$, it follows that $a_1 \in Q$. Since $(P, Q) \in R_A$ and $P \cap (a_1 \Rightarrow a_2) \neq \emptyset$ we conclude that $a_2 \in Q$. Thus, $Q \in \beta(a_2) \cap Y$. Now, since $(a_2, b_2) \in \Theta(Y)$ it follows that $b_2 \in Q$ which is a contradiction. So, $\bigcup \beta[a_1 \Rightarrow a_2] \cap Y \subseteq \beta[b_1 \Rightarrow b_2] \cap Y$. The other inclusion can be proved in a similar way. ☐

Theorem 34. *Let* $\mathbf{A} \in \mathsf{DLatGi}$. *Then the map* Θ *establishes a dual isomorphism between the set of closed and* $R_{\mathbf{A}}$*-closed subsets of* $X(\mathbf{A})$ *ordered by inclusion and the set of the gi-congruences of* $\mathbf{A}$*, also ordered by inclusion.*

Proof. It follows from the last lemma and the dual isomorphism between the lattice of congruences of a bounded distributive lattice and the lattice of closed sets of its dual Priestley space. ☐

Corollary 35. *Let* $\mathbf{A} \in \mathsf{DLatGi}$. *Then the set of the gi-congruences of* $\mathbf{A}$ *ordered by inclusion is a complete lattice.*

Proof. It follows from the theorem and the fact that the set of the closed and $R_{\mathbf{A}}$-closed subsets of $X(\mathbf{A})$ ordered by inclusion is a complete lattice because the intersection of every family of $R_{\mathbf{A}}$-closed sets is $R_{\mathbf{A}}$-closed. ☐

We show that the lattice of gi-congruences of every $\mathbf{A} \in \mathsf{DLatGi}$ is indeed a sublattice of the lattice of congruences of A.

Proposition 36. *Let* $\mathbf{A} \in \mathsf{DLatGi}$. *Then* $\mathrm{gCon}\mathbf{A}$ *is a sublattice of the lattice* $\mathrm{Con}A$*, with the same bounds.*

Proof. Let θ_1 and θ_2 be two gi-congruences of $\mathbf{A}$. We know that $\theta_1 \cap \theta_2$ is a congruence on A. Let $(a, b), (c, d) \in \theta_1 \cap \theta_2$. We prove that $\langle a \Rightarrow c, b \Rightarrow d\rangle \in \theta_1 \cap \theta_2$. To prove condition 1 of Definition 7, let $x \in (a \Rightarrow c)$. Since θ_1 and θ_2 are gi-congruences, there are $y_1, y_2 \in (b \Rightarrow d)$ such that $(x, y_1) \in \theta_1$ and $(x, y_2) \in \theta_2$. Let $y = y_1 \vee y_2$. Then, a not too difficult computation shows that $(x, y) \in \theta_1 \cap \theta_2$. Therefore, since $y \in (b \Rightarrow d)$, condition 1 of Definition 7 follows. Condition 2 of Definition 7 is proved similarly. Consequently, $\theta_1 \cap \theta_2$ is a gi-congruence.

To prove that $\theta_1 \vee \theta_2$ is a gi-congruence we use duality. Let $(a, b), (c, d) \in \theta_1 \vee \theta_2$. We show that $\langle a \Rightarrow c, b \Rightarrow d\rangle \in \theta_1 \vee \theta_2$. We know that $\theta_1 \vee \theta_2$ is a congruence of A. We consider the closed sets Y_1, Y_2 of $X(A)$ associated with θ_1 and θ_2 respectively. By the isomorphism between the lattice of closed sets and the lattice of congruences, $Y_1 \cap Y_2$ is the closed set associated with $\theta_1 \vee \theta_2$, that is $\theta(Y_1 \cap Y_2) = \theta_1 \vee \theta_2$. Thus, to show that $\theta_1 \vee \theta_2$ is a congruence of A we need to prove that $\langle a \Rightarrow c, b \Rightarrow d\rangle \in \theta(Y_1 \cap Y_2)$. According to Lemma 32, it is enough to prove that

$$\bigcup \beta[a \Rightarrow c] \cap (Y_1 \cap Y_2) = \bigcup \beta[b \Rightarrow d] \cap (Y_1 \cap Y_2).$$

Let $P \in \bigcup \beta[a \Rightarrow c] \cap (Y_1 \cap Y_2)$. Thus, $P \cap (a \Rightarrow c) \neq \emptyset$ and $P \in Y_1 \cap Y_2$. Suppose that $P \notin \bigcup \beta[b \Rightarrow d]$ Then, $P \cap (b \Rightarrow d) = \emptyset$. By Lemma 15 there exists $Q \in X(\mathbf{A})$ such that $(P, Q) \in R_A$, $b \in Q$ and $d \notin Q$. Since $P \in Y_1 \cap Y_2$, by the above lemma, $R_A(P) \subseteq Y_1 \cap Y_2$. Thus, $Q \in Y_1 \cap Y_2$. Moreover, since $b \in Q$, $Q \in \beta(b) \cap Y_1 \cap Y_2$. Now, since $(a, b) \in \theta(Y_1 \cap Y_2)$ it follows that $a \in Q$. From the facts that $P \cap (a \Rightarrow c) \neq \emptyset$ and $(P, Q) \in R_A$ it then follows that $c \in Q$. So, $Q \in \beta(c) \cap Y_1 \cap Y_2$ and since $(c, d) \in \theta(Y_1 \cap Y_2)$ we have $d \in Q$, which is a contradiction. Therefore we have obtained that $\bigcup \beta[a \Rightarrow c] \cap (Y_1 \cap Y_2) \subseteq \bigcup \beta[b \Rightarrow d] \cap (Y_1 \cap Y_2)$. The other inclusion is proved analogously. ☐

In every bounded distributive lattice A the lattice of filters is dually isomorphic to the lattice of closed up-sets of its dual Priestley space $X(A)$. We characterize the filters of a gi-lattice $\mathbf{A}$ that correspond to the closed up-sets whose associated congruence is a gi-congruence, namely the filters that correspond to the closed up-sets which are R_A-closed.

Let A be a bounded distributive lattice. We denote by $\mathcal{C}_u(X(A))$ the closed up-sets of it dual Priestley space. Let $\psi : \mathrm{Fi}(A) \to \mathcal{C}_u(X(A))$ be the map given by

$$\psi(F) = \{P \in X(A) : F \subseteq P\}.$$

This map establishes the dual isomorphism between $\mathrm{Fi}(A)$ and $\mathcal{C}_u(X(A))$.

Proposition 37. *Let* $\mathbf{A} \in \mathsf{DLatGi}$ *and* F *a filter of* A*. Then* $\psi(F)$ *is* R_A*-closed if and only if for every* $a \in F$, $F \cap (1 \Rightarrow a) \neq \emptyset$.

Proof. Suppose that $\psi(F)$ is R_A-closed. Let $a \in F$ and assume that $F \cap 1 \Rightarrow a = \emptyset$. Let $P \in X(A)$ be such that $F \subseteq P$ and $(1 \Rightarrow a) \cap P = \emptyset$. Then, there is $Q \in X(A)$ such that $(P, Q) \in R_A$, $a \notin Q$ and $F \subseteq P$. Therefore $P \in \psi(F)$ and, since $\psi(F)$ is R_A-closed, we have $Q \in \psi(F)$, so $a \in Q$, a contradiction.

Suppose now that for every $a \in F$, $F \cap (1 \Rightarrow a) \neq \emptyset$. Let $P \in \psi(F)$ and $Q \in X(A)$ be such that $(P, Q) \in R_A$. Suppose $a \in F$. Then $F \cap 1 \Rightarrow a \neq \emptyset$, and so $P \cap (1 \Rightarrow a) \neq \emptyset$. Since $(P, Q) \in R_A$ and $1 \in Q$ it follows that $a \in Q$. Therefore $F \subseteq Q$ and so, $Q \in \psi(F)$. ☐

This result suggests a stronger notion of filter for gi-lattices. Let $\mathbf{A} \in \mathsf{DLatGi}$. A set $F \subseteq A$ is a *gi-filter* of $\mathbf{A}$ if it is a lattice filter of A and for every $a \in A$, $F \cap 1 \Rightarrow a \neq \emptyset$. Thus Proposition 37 gives:

Proposition 38. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. The ordered set of the gi-filters of* $\mathbf{A}$ *is dually isomorphic to the ordered set of the closed up-sets of* $X(\mathbf{A})$ *that are* R_A*-closed, and hence it is a complete lattice.*

Theorem 34 and Proposition 38 imply that the lattice of gi-filters is isomorphic to a sublattice of the lattice of gi-congruences. We characterize the gi-congruences that correspond to gi-filters.

Let $\mathbf{A} \in \mathsf{DLatGi}$. A congruence θ of A is *compatible* with a prime filter P of A if for every $a, b \in A$ such that $a\theta b$ and $a \in P$, then $b \in P$. We say that θ is *increasing* if for all prime filters $P, Q \in X(A)$, if θ is compatible with P and $P \subseteq Q$, then θ is compatible with Q.

It is a well-known fact of Priestley duality for distributive lattices that for every bounded distributive lattice A and every congruence θ of A,

$$\Theta^{-1}(\theta) = \{P \in X(A) : \theta \text{ is compatible with } P\}.$$

Therefore, $\Theta^{-1}(\theta)$ is a closed up-set of the Priestley space $X(A)$ if and only if θ is increasing. This fact, together with Proposition 38 and Theorem 34, implies:

Theorem 39. *Let* $\mathbf{A} \in \mathsf{DLatGi}$*. The lattice of gi-filters of* $\mathbf{A}$ *is dually isomorphic to the lattice of increasing gi-congruences.*

7. Concluding remarks

The distributive lattices with a generalized implication are not algebras in the Universal algebra sense, but we have been able to study them as if they were. For example, as we have seen, we have a notion of congruence. The corresponding notion to that of subalgebra can also be defined for distributive lattices with a generalized implication and appropriate concepts of subdirectly irreducible and simple gi-lattices can also be defined. In [2] the analogous notions were introduced for Quasi-modal algebras. These concepts will be analyzed in a sequel to this paper.

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