Abstract Algebraic Logic
An overview (III)

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Some non-protoalgebraic logics

> *"The class of protoalgebraic logics includes all the non-pathological propositional logics considered in the literature we are aware of."*
>
> [BLOK and PIGOZZI, 1986]

- The fragments of $\mathcal{CPL}$ or $\mathcal{IPL}$ with $\wedge , \vee$ (and also those with $\top , \bot$).
- $\mathcal{IPL}^*$, the fragment of $\mathcal{IPL}$ with $\neg , \wedge , \vee$ (and also that with $\top , \bot$).
- BELNAP's 4-valued logic, with $\neg , \wedge , \vee$ (and also if with $\top , \bot$).
- The weak relevance logic of WÓJCICKI.
- The infinite-valued logic $Ł_\infty^\leqslant$ determined by $\leqslant$ in $[\mathbf{0{,}1}]$.
- Many *fuzzy* and *substructural* logics preserving degrees of truth.
- The $\Box$-fragments of normal modal logics.
- Some *positive* modal logics (the $\langle \to , \neg \rangle$-free fragments).
- Some *subintuitionistic* logics.

For these logics, $\mathbf{Alg}^*\mathcal{L}$ is not guaranteed to be what it **should** be

Definition (RAFTERY)

A logic $\mathcal{L}$ is **truth-equational** when for every $\langle A, F\rangle \in \mathbf{Mod}^*\mathcal{L}$, the filter F is equationally definable: $F = \{a \in A : \boldsymbol{A} \models E(x)\,[\![a]\!]\}$ for some $E(x) \subseteq Fm \times Fm$.

- $\{$weakly algebraizable$\} \subsetneq \{$truth-equational$\} \subsetneq \{$has algebraic semantics$\}$
- Weakly algebraizable $\iff$ protoalgebraic + truth-equational.
- Algebraizable $\iff$ equivalential + truth-equational.

  (here truth-equational cannot be weakened to has algebraic semantics.)
- If $\mathcal{L}$ is protoalgebraic, then:

  $\mathcal{L}$ is truth-equational $\iff$ $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is injective on $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every $\boldsymbol{A}$.
- For arbitrary $\mathcal{L}$, $\Rightarrow$ holds, but $\Leftarrow$ needs not hold.
- $\mathcal{L}$ is truth-equational $\iff$ $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is completely order-reflecting on $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ for every $\boldsymbol{A}$.
- **Main tool**: Study of **the Suszko operator** [CZELAKOWSKI , RAFTERY]

$$F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \quad \longmapsto \quad \widetilde{\boldsymbol{\Omega}}{}^{\mathcal{L}}_{\boldsymbol{A}}F = \bigcap\{\boldsymbol{\Omega}_{\boldsymbol{A}}G : G \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}\,,\ G \supseteq F\}$$

The semantics of generalized matrices

- A **generalized matrix** is $\langle A, \mathcal{C}\rangle$ with $\mathcal{C} \subseteq P(A)$ closed under $\bigcap$, and $A \in \mathcal{C}$.
- $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a **generalized model of** $\mathcal{L}$ when $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$. **GMod**$\mathcal{L}$.
- $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a **generalized model of** $\mathcal{L}$ when its associated closure operator C satisfies: if $\Gamma \vdash_{\mathcal{L}} \varphi$ then $v(\varphi) \in C\big(v[\Gamma]\big)$ $\forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$.
- A g-model of $\mathcal{L}$ is **basic full** when it has the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}\rangle$.
- The Tarski operator: $\mathcal{C} \longmapsto \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\mathcal{C} = \bigcap\big\{\boldsymbol{\Omega}_{\boldsymbol{A}}G : G \in \mathcal{C}\big\}$
- $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is reduced when $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}\mathcal{C} = Id_A$. **GMod**$^*\mathcal{L}$, etc.

Theorem

Every $\mathcal{L}$ is complete with respect to:

- **GMod**$\mathcal{L}$
- **GMod**$^*\mathcal{L}$
- $\big\{$*basic full g-models of* $\mathcal{L}\big\}$
- $\big\{$*reduced basic full g-models of* $\mathcal{L}\big\}$

The truly general definition of $\mathcal{L}$-algebras

Definition (Font and Jansana, 1996)

$\mathbf{Alg}\mathcal{L}$ is the class of algebra reducts of reduced g-models of $\mathcal{L}$.

$$\mathbf{Alg}\mathcal{L} = \{\boldsymbol{A} : \exists\, \mathcal{C} \subseteq \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \text{ such that } \langle \boldsymbol{A}, \mathcal{C} \rangle \text{ is reduced}\}.$$

- $\boldsymbol{A} \in \mathbf{Alg}\mathcal{L} \iff \langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \rangle$ is reduced.
- $\mathbf{Alg}\mathcal{L} = \mathbb{P}_{SD}(\mathbf{Alg}^{*}\mathcal{L})$.
- $\mathcal{L}$ protoalgebraic $\Rightarrow$ $\mathbf{Alg}\mathcal{L} = \mathbf{Alg}^{*}\mathcal{L}$. ($\not\Leftarrow$)
- $\mathcal{L}$ algebraizable $\Rightarrow$ $\mathbf{Alg}\mathcal{L}$ coincides with its equivalent algebraic semantics.
- $\mathbf{Alg}\mathcal{CPL}_{\wedge\vee} = \{\text{(bounded) distributive lattices}\} \supsetneq \mathbf{Alg}^{*}\mathcal{CPL}_{\wedge\vee}$.
- $\mathbf{Alg}(\mathcal{B}elnap) = \{\text{De Morgan algebras}\} \supsetneq \mathbf{Alg}^{*}(\mathcal{B}elnap)$.
- $\mathbf{Alg}Ł_{\infty}^{\leqslant} = \{\text{Wajsberg algebras (MV-algebras)}\} = \mathbf{Alg}^{*}Ł_{\infty}^{\leqslant} = \mathbf{Alg}Ł_{\infty}$.

Full g-models

Definition

A g-model $\langle A, \mathcal{C} \rangle$ of $\mathcal{L}$ is **full** when its reduction is basic full,

i.e., when $\mathcal{C} / \widetilde{\Omega}_A(\mathcal{C}) = \mathcal{F}i_{\mathcal{L}}\big(A / \widetilde{\Omega}_A(\mathcal{C})\big)$.

- Every logic $\mathcal{L}$ is complete wrt the classes **FGMod**$\mathcal{L}$ and **FGMod**$^*\mathcal{L}$.
- **Alg**$\mathcal{L}$ is the class of algebra reducts of reduced full g-models of $\mathcal{L}$.

Theorem (THE ISOMORPHISM THEOREM)

$\forall$ *finitary* $\mathcal{L}$, $\forall A$, $\quad \widetilde{\Omega}_A : \{$ *full g-models of* $\mathcal{L}$ *on* $A \} \cong \mathrm{Co}_{\mathbf{Alg}\mathcal{L}} A$.

- $\langle A, \mathcal{C} \rangle \in$ **FGMod**$^*\mathcal{L} \iff A \in$ **Alg**$\mathcal{L}$ and $\mathcal{C} = \mathcal{F}i_{\mathcal{L}} A$.

G-matrices as models of Gentzen systems

Definitions

A g-matrix $\langle \boldsymbol{A}, C \rangle$ is a **model of a Gentzen-style rule**

$$\frac{\{\Gamma_i \rhd \varphi_i : i < n\}}{\Gamma \rhd \varphi}$$

when $\forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $v(\varphi_i) \in C\big(v[\Gamma_i]\big)$ $\forall i < n$
then $v(\varphi) \in C\big(v[\Gamma]\big)$.

A g-matrix $\langle \boldsymbol{A}, C \rangle$ is a **model of a Gentzen system**
when it is a model of all its derivable rules.

Mod$\mathfrak{G}$, **Mod**$^*\mathfrak{G}$, **Alg**$\mathfrak{G}$ as usual.

G-matrices as models of Gentzen systems

Definition

A Gentzen system $\mathfrak{G}$ is **fully adequate** for a sentential logic $\mathcal{L}$ when the models of $\mathfrak{G}$ coincide with the full g-models of $\mathcal{L}$.

- A fully adequate Gentzen system need not exist for a given logic.
- If it exists, it is unique.
- It is unique among all the Gentzen systems $\mathfrak{G}$ that are **adequate** for a sentential logic $\mathcal{L}$, i.e., such that for finite Γ,

  $$\Gamma \vdash_{\mathcal{L}} \varphi \iff \text{the sequent } \Gamma \rhd \varphi \text{ is derivable in } \mathfrak{G}.$$
- If $\mathfrak{G}$ is fully adequate for $\mathcal{L}$ then:
  - $\mathcal{L}$ is the weakest model of $\mathfrak{G}$: the rules of $\mathfrak{G}$ characterize $\mathcal{L}$.
  - $A \in \mathbf{Alg}\mathcal{L} \iff \langle A, \mathcal{F}i_{\mathcal{L}}A \rangle$ is a reduced model of $\mathfrak{G} \iff A \in \mathbf{Alg}\mathfrak{G}$.
- If $\mathcal{L}$ is finitely algebraizable, then $\mathcal{L}$ has a fully adequate Gentzen system if and only if $\mathcal{L}$ satisfies the Deduction Theorem.

The Frege hierarchy

Definitions

- $\mathcal{L}$ is **selfextensional** when it has the weak replacement property:

$$\alpha \dashv\vdash_{\mathcal{L}} \beta \;\Rightarrow\; \varphi(\alpha) \dashv\vdash_{\mathcal{L}} \varphi(\beta) \quad \forall \varphi \in Fm$$

  i.e., when the relation $\alpha \equiv \beta \; (\boldsymbol{\Lambda}\mathcal{L}) \Leftrightarrow \alpha \dashv\vdash_{\mathcal{L}} \beta$, is a congruence of $\boldsymbol{Fm}$.

- $\mathcal{L}$ is **Fregean** when it has the strong replacement property: $\forall \Gamma \subseteq Fm$:

$$\left.\begin{array}{l} \Gamma, \alpha \vdash_{\mathcal{L}} \beta \\ \Gamma, \beta \vdash_{\mathcal{L}} \alpha \end{array}\right\} \Rightarrow \; \Gamma, \varphi(\alpha) \vdash_{\mathcal{L}} \varphi(\beta) \quad \forall \varphi \in Fm.$$

  Equivalently, when for every theory T of $\mathcal{L}$, the relation

$$\alpha \equiv \beta \; (\boldsymbol{\Lambda}_{\mathcal{L}} T) \Leftrightarrow T, \alpha \vdash_{\mathcal{L}} \beta \text{ and } T, \beta \vdash_{\mathcal{L}} \alpha$$

  is a congruence of $\boldsymbol{Fm}$.

The Frege hierarchy

Definitions

- $\mathcal{L}$ is **fully selfextensional** when all its full g-models $\langle \boldsymbol{A}, C \rangle$ have the congruence property, i.e., the relation

  $$a \equiv b \ (\boldsymbol{\Lambda} C) \Leftrightarrow C(a) = C(b)$$

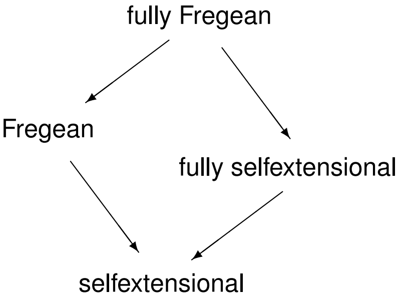
  is a congruence of $\boldsymbol{A}$.

- $\mathcal{L}$ is **fully Fregean** when all axiomatic extensions of all its full g-models $\langle \boldsymbol{A}, C \rangle$ have the congruence property, i.e., for every $T \in \mathcal{C}$, the relation

  $$a \equiv b \ (\boldsymbol{\Lambda}_C T) \Leftrightarrow C\big(T, a\big) = C(T, b)$$

  is a congruence of $\boldsymbol{A}$.

The Frege hierarchy


fully Fregean

Fregean

fully selfextensional

selfextensional

- The four classes are distinct.
- **Does Fregean + fully selfextensional $\Rightarrow$ fully Fregean ?**
- **$\exists\, \mathcal{L}$ that is selfextensional but neither fully selfextensional nor Fregean ?**
- If $\mathcal{L}$ has the “uniterm” DDT or a Conjunction, then

$$\mathcal{L} \text{ is selfextensional} \iff \mathcal{L} \text{ is fully selfextensional.}$$

Interplay between the two hierarchies

- Selfextensionality seems to be rather independent of the Leibniz hierarchy.

	selfextensional	non-selfextensional
algeb.	$\mathcal{IPL}, \mathcal{CPL}$ and fragments with $\to / \leftrightarrow$	$\mathcal{BCK}, Ł_\infty, \mathcal{R}, \mathcal{RM}$ global $\mathcal{K}, \mathcal{T}, \mathcal{S}4, \mathcal{S}5$
proto.	local $\mathcal{K}, \mathcal{T}, \mathcal{S}4, \mathcal{S}5$	relevance $E, T, \mathcal{C}1$, ortholattices
non-proto.	$\mathcal{CPL}_{\wedge\vee}, \mathcal{IPL}^*, \mathcal{B}elnap, Ł_\infty^{\leqslant}$	PRIEST's logic; *ad hoc* examples

- If $\mathcal{L}$ is protoalgebraic, then $\mathcal{L}$ is Fregean $\Leftrightarrow$ $\mathcal{L}$ is fully Fregean.
- If $\mathcal{L}$ is weakly algebraizable, then $\mathcal{L}$ is Fregean $\Leftrightarrow$ $\mathcal{L}$ is fully selfextensional.
- If $\mathcal{L}$ is fully selfextensional then $\mathcal{L}$ is algebraizable $\Leftrightarrow$ $\mathcal{L}$ is weakly algebraizable.
- If $\mathcal{L}$ is Fregean and not almost inconsistent, then $\mathcal{L}$ is protoalgebraic $\Rightarrow$ $\mathcal{L}$ is regularly algebraizable.
- **Does selfextensional + protoalgebraic/algebraizable $\Rightarrow$ fully selfextensional ?**

Algebraizability as based on structural translations

Recall:

$$\begin{tikzcd}\mathcal{L} \arrow[r, shift left, "\boldsymbol{\tau}"] & \mathbf{K} \arrow[l, shift left, "\boldsymbol{\rho}"]\end{tikzcd}$$

$$\alpha \longrightarrow \boldsymbol{\tau}(\alpha) = E(\alpha)$$ *defining equations* $E(x)$

$$\Delta(\delta,\varepsilon) = \boldsymbol{\rho}(\delta \approx \varepsilon) \longleftarrow \delta \approx \varepsilon$$ *equivalence formulas* $\Delta(x,y)$

The translations are **structural** $\iff$ the translations are defined from sets $E(x)$ and $\Delta(x,y)$ respectively.

$\iff$ the translations commute with substitutions:

$\boldsymbol{\tau}\big(\sigma(\alpha)\big) = \sigma\big[\boldsymbol{\tau}(\alpha)\big]$ and

$\boldsymbol{\rho}\big(\sigma(\delta) \approx \sigma(\varepsilon)\big) = \sigma\big[\boldsymbol{\rho}(\delta \approx \varepsilon)\big]$

Structural equivalence of other "deductive structures"

$$\langle Form\,,\mid\!\sim\rangle = \mathfrak{G} \underset{\boldsymbol{\rho}}{\overset{\boldsymbol{\tau}}{\rightleftarrows}} \mathfrak{G}' = \langle Form'\,,\mid\!\sim'\rangle \quad \text{with (A1)–(A4)}$$

Each set of "formulas" $Form\,, Form'$ can be of a different *kind*.

$m = 1$: $Form = Fm$, $\mid\!\sim \, = \, \vdash_{\mathcal{L}}$ (sentential logic)

$m = 2$: $Form' = Fm \times Fm$, $\mid\!\sim' \, = \, \models_{\mathbf{K}}$ (relative equational consequence)

- BLOK, PIGOZZI [1991,1992] m-dimensional deductive systems.
- Gentzen systems.
- Many-sided Gentzen systems.
- Hypersequents.

$\mathfrak{G}$ is **algebraizable** $\overset{\text{def}}{\Longleftrightarrow}$ $\mathfrak{G}$ is structurally equivalent to $\langle Fm \times Fm, \models_{\mathbf{K}}\rangle$.
Then **K** is its **equivalent algebraic semantics**

Abstract, all-encompassing theory: BLOK and JÓNSSON [1999 , 2006].

"Logification" of classes of algebras (the example of **DL**)

- **DL** is not the equivalent algebraic semantics of any algebraizable logic.
- $\mathcal{CPL}_{\wedge\vee}$ is non-protoalgebraic.
- **DL** is the equivalent algebraic semantics of the Gentzen system with all structural rules and the usual rules for conjunction and disjunction.
- This Gentzen system is algebraizable, with the translations:
  $\boldsymbol{\tau}(\alpha_1,\dots,\alpha_n \rhd \beta) = \{\alpha_1 \wedge \cdots \wedge \alpha_n \preccurlyeq \beta\}$, $\boldsymbol{\rho}(\alpha \approx \beta) = \{\alpha \rhd \beta, \beta \rhd \alpha\}$.
- This Gentzen system is fully adequate for $\mathcal{CPL}_{\wedge\vee}$.
- $\langle \boldsymbol{A}, C\rangle$ is a full g-model of $\mathcal{CPL}_{\wedge\vee}$ $\iff$ is a model of this Gentzen system
  $\iff C(\varnothing) = \varnothing$, $C(a \wedge b) = C(a,b)$, $C(X, a \vee b) = C(X,a) \cap C(X,b)$.
- $\mathcal{CPL}_{\wedge\vee}$ is fully Fregean.
- $\forall \boldsymbol{A} \in$ **DL**, $\mathrm{Co}\boldsymbol{A} \cong \{\mathcal{C} \subseteq P(A) : \langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full g-model of $\mathcal{CPL}_{\wedge\vee}\}$.

Logics without implication

A new kind of “algebraizability” for sentential logics

Definition

A logic $\mathcal{L}$ is **G-algebraizable** wrt a class of algebras $\mathbf{K}$ when there is a Gentzen system $\mathfrak{G}$ such that:

- $\mathfrak{G}$ is fully adequate for $\mathcal{L}$.
- $\mathfrak{G}$ is algebraizable and $\mathbf{K}$ is its equivalent algebraic semantics.

(Then $\mathbf{Alg}\mathcal{L} = \mathbf{Alg}\mathfrak{G} = \mathbf{K}$)

- Algebraizable but not G-algebraizable:

  $\mathcal{BCK}$, $\mathcal{R}$, $Ł_\infty$, the global consequences of modal $\mathcal{K}$, $\mathcal{T}$.

- G-algebraizable but not algebraizable:

  $\mathcal{CPL}_{\wedge\vee}$, $\mathcal{B}elnap$, $Ł_\infty^{\leqslant}$, etc. Actually:

Theorem (Font and Jansana, 1996)

If $\mathcal{L}$ is selfextensional and satisfies the “uniterm” DDT or has a Conjunction, then $\mathcal{L}$ is G-algebraizable.

The variety problem

Why is Alg$\mathcal{L}$ a variety so often ?

- For algebraizable logics, the general theory only guarantees an $\mathbb{SP}$-class.
- In the finitary case, the general theory only guarantees a quasivariety.
- Few examples are known where $\mathcal{L}$ is a proper quasivariety:

  $\mathcal{BCK}$, $\mathcal{P}1$, $\mathcal{IPL}_{\leftrightarrow,\neg}$, the weakest implicative logic.
- For many non-protoalgebraic $\mathcal{L}$, **Alg**$\mathcal{L}$ is also a variety.
- Theorem: If $\mathcal{L}$ is selfextensional and has the “uniterm” DDT or a Conjunction, then **Alg**$\mathcal{L}$ is a variety.
- But: Fregean + regularly algebraizable $\not\Longrightarrow$ **Alg**$\mathcal{L}$ is a variety.
- **Other conditions ? Characterizations of strong algebraizability ?**