Abstract Algebraic Logic
An overview (II)

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Master in Pure and Applied Logic
2007–2008

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Matrix semantics

A **logical matrix** is $\langle \boldsymbol{A}, F \rangle$ with $F \subseteq A$. F is the **filter**/**truth set** of the matrix.

Want a class **M** of matrices **complete** for $\mathcal{L}$; i.e., such that

$$(1) \qquad \Gamma \vdash_{\mathcal{L}} \varphi \iff \forall \langle \boldsymbol{A}, F \rangle \in \mathbf{M}, \ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \text{ if } v(\gamma) \in F \ \forall \gamma \in \Gamma \text{ then } v(\varphi) \in F.$$

Definition

$\langle \boldsymbol{A}, F \rangle$ is a **model** of $\mathcal{L}$ when it satisfies $(\Rightarrow)$ in (1), i.e.,

$$\Gamma \vdash_{\mathcal{L}} \varphi \implies \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \text{ if } v(\gamma) \in F \ \forall \gamma \in \Gamma \text{ then } v(\varphi) \in F.$$

F is a **filter of** $\mathcal{L}$ when $\langle \boldsymbol{A}, F \rangle$ is a model of $\mathcal{L}$.

$$\mathbf{Mod}\mathcal{L} = \{\text{models of } \mathcal{L}\}.$$

For each $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A} = \{F \subseteq A : F \text{ is a filter of } \mathcal{L} \text{ over } \boldsymbol{A}\}$.

Matrix semantics

Theorem (Lindenbaum, Wójcicki)

For every logic $\mathcal{L}$, the class **Mod**$\mathcal{L}$ *is a complete matrix semantics for $\mathcal{L}$.*

Proof: Lindenbaum matrices $\{\langle \boldsymbol{Fm}, T\rangle : T \text{ a theory of } \mathcal{L}\} \subseteq$ **Mod**$\mathcal{L}$. ∎

- **More meaningful solutions: Reduced matrices**

$$\forall \langle \boldsymbol{A}, F\rangle \quad \exists\, \boldsymbol{\Omega}_{\boldsymbol{A}} F = \max\{\theta \in \mathrm{Co}\boldsymbol{A} : a \in F,\ a \equiv b\ (\theta) \Rightarrow b \in F\}$$

the **Leibniz congruence** of $\langle \boldsymbol{A}, F\rangle$

(it is an intrinsic, algebraic property of $\boldsymbol{A}$ and F.)

reduced models: **Mod**$^*\mathcal{L} = \{\langle \boldsymbol{A}, F\rangle \in$ **Mod**$\mathcal{L} : \boldsymbol{\Omega}_{\boldsymbol{A}} F = Id_A\}$

$\mathcal{L}$-algebras: **Alg**$^*\mathcal{L} = \{\boldsymbol{A} : \exists F \subseteq A \text{ with } \langle \boldsymbol{A}, F\rangle \in$ **Mod**$^*\mathcal{L}\}$

$\langle \boldsymbol{A}, F\rangle \in$ **Mod**$\mathcal{L}$ $\longmapsto$ $\langle \boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}} F, F/\boldsymbol{\Omega}_{\boldsymbol{A}} F\rangle \in$ **Mod**$^*\mathcal{L}$

$\boldsymbol{A}/\boldsymbol{\Omega}_{\boldsymbol{A}} F \in$ **Alg**$^*\mathcal{L}$

The LINDENBAUM-TARSKI process, generalized

Assume $\Gamma \nvdash_{\mathcal{L}} \varphi$.

Let T be the $\mathcal{L}$**-theory generated by** Γ. Then $\langle \boldsymbol{Fm}, T \rangle \in \mathbf{Mod}\mathcal{L}$.

(2) *Take* $\boldsymbol{\Omega}_{\boldsymbol{Fm}} T$.

We know that:

(3) $\boldsymbol{\Omega}_{\boldsymbol{Fm}} T$ is a **congruence** of the formula algebra $\boldsymbol{Fm}$.

(4) The quotient matrix $\langle \boldsymbol{Fm} / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T, T / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T \rangle \in \mathbf{Mod}^{*}\mathcal{L}$.

(5) $\alpha \in T \iff \alpha / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T \in T / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T$.

Take $\boldsymbol{A} := \boldsymbol{Fm} / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T$, $F := T / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T$, $v(x) := x / \boldsymbol{\Omega}_{\boldsymbol{Fm}} T \quad \forall x \in Var$.

This shows:

Theorem (WÓJCICKI)

For every logic $\mathcal{L}$*, the class* $\mathbf{Mod}^{*}\mathcal{L}$ *is a complete matrix semantics for* $\mathcal{L}$.

The Leibniz operator

- On each algebra A, we could consider
$$\begin{array}{rccc} \Omega_A : & P(A) & \longrightarrow & \mathrm{Co}A \\ & F & \longmapsto & \Omega_A F \end{array}$$

Let **K** be a class of algebras and A be any algebra.

We consider the **relative congruences**: $\mathrm{Co}_{\mathbf{K}}A = \{\theta \in \mathrm{Co}A : A/\theta \in \mathbf{K}\}$.

If $F \in \mathcal{F}i_{\mathcal{L}}A$, then $A/\Omega_A F \in \mathbf{Alg}^*\mathcal{L}$, i.e., $\Omega_A F \in \mathrm{Co}_{\mathbf{Alg}^*\mathcal{L}}A$.

- So we restrict:
$$\begin{array}{rccc} \Omega_A : & \mathcal{F}i_{\mathcal{L}}A & \longrightarrow & \mathrm{Co}_{\mathbf{Alg}^*\mathcal{L}}A \\ & F & \longmapsto & \Omega_A F \end{array}$$
(always surjective)

The behaviour of this operator on the filters of a logic determines the "algebraic behaviour" of the logic

Theorem (main intrinsic characterization)

*A logic $\mathcal{L}$ is **algebraizable***

if and only if

the Leibniz operator on the formula algebra $\boldsymbol{\Omega}_{\boldsymbol{Fm}}$

is monotone, injective and commutes with inverse substitutions.

(commutes with inverse substitutions: $\boldsymbol{\Omega}_{\boldsymbol{Fm}}\sigma^{-1}[T] = \sigma^{-1}\big[\boldsymbol{\Omega}_{\boldsymbol{Fm}}T\big]$ for all T, σ.)

(monotone + injective $\Rightarrow$ $\boldsymbol{\Omega}_{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{L} \cong \mathrm{Co}_{\mathbf{Alg}^*\mathcal{L}}\boldsymbol{Fm}$)

Theorem (The isomorphism theorem)

Let $\mathcal{L}$ be a finitary logic, and let **K** *be a quasivariety.*

*Then $\mathcal{L}$ is **algebraizable** and* **K** *is its equivalent algebraic semantics*

if and only if

for every algebra $\boldsymbol{A}$, $\boldsymbol{\Omega}_{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \cong \mathrm{Co}_{\mathbf{K}}\boldsymbol{A}$.

Applications (I)

- Algebraizability is an intrinsic property of a logic.
- Helps to show that a certain $\mathcal{L}$ is algebraizable, once we empirically know **K**, and to confirm that **K** is the equivalent algebraic semantics.
  - ANDERSON and BELNAP's relevance logic $\mathcal{R}$.
  - ŁUKASIEWICZ's $Ł_\infty$ $\cdots$ Wajsberg algebras (MV-algebras).
- Helps to show that a logic is not algebraizable, wrt to any **K**, $E(x)$, $\Delta(x,y)$.
  - $\mathcal{IPL}^*$ ($\mathcal{IPL}$ without implication).
  - Relevance implication $\mathcal{R}_{\rightarrow}$.
  - DA COSTA's paraconsistent logic $\mathcal{C}1$.
  - The local consequences associated with normal modal logics.
- Helps to show that some **K** is not the equivalent algebraic semantics of any algebraizable logic. ("**K** is not logifiable.")
  - $\{$distributive lattices$\}$ and $\{$De Morgan algebras$\}$ are not "logifiable".
- If a (quasi)-variety **K** satisfies some kind of isomorphism theorems, maybe there is some algebraizable $\mathcal{L}$ whose equivalent algebraic semantics is **K**.

Applications (II)

- $\mathbf{K}$ is an equivalent algebraic semantics of an algebraizable logic if and only if there are $E(x) \subseteq Fm \times Fm$ and $\Delta(x,y) \subseteq Fm$ such that:

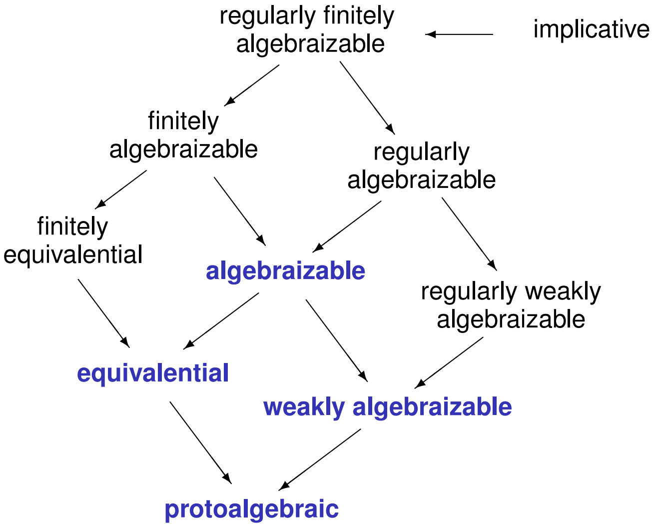
$$\left.\begin{array}{l} \mathbf{K} \models E\big(\Delta(x,x)\big) \\ E\big(\Delta(x,y)\big) \models_{\mathbf{K}} x \approx y \end{array}\right\} \qquad \text{(A4)}$$

- $\mathcal{L}$ is determined by the completeness condition (1), i.e., (A1):

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \forall \boldsymbol{A} \in \mathbf{K}\,,\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\,,$$
$$\text{if} \quad \boldsymbol{A} \models E(x)\,[\![v(\gamma)]\!]\ \ \forall \gamma \in \Gamma \quad \text{then} \quad \boldsymbol{A} \models E(x)\,[\![v(\varphi)]\!].$$

- If $\mathbf{K}$ is the (quasi-)variety generated by a single algebra, then it is enough to check (A4) for this algebra.
  - MOISIL's "determination principle" in 3-valued Łukasiewicz algebras: If $a \to b = b \to a = \neg a \to \neg b = \neg b \to \neg a = 1$ then $a = b$.
  - SETTE's "maximal paraconsistent logic" $\mathcal{P}1$ is algebraizable, with $\Delta(x,y) = \{x \to y\,, y \to x\,, \neg x \to \neg y\,, \neg y \to \neg x\}$ and $E(x) = \{\top \to x \approx \top\}$.

(a part of) the Leibniz (or protoalgebraic) hierarchy



Syntactic characterizations

- $\mathcal{L}$ is **protoalgebraic** when $\exists\, \Delta(x,y) \subseteq Fm$ such that

  (3a) $\quad \vdash_{\mathcal{L}} \Delta(\alpha,\alpha)$

  (MP) $\quad \alpha\,, \Delta(\alpha,\beta) \vdash_{\mathcal{L}} \beta$

  (e.g., all logics with $\vdash_{\mathcal{L}} \alpha \to \alpha$ and $\alpha\,, \alpha \to \beta \vdash_{\mathcal{L}} \beta$)

  (there is only one protoalgebraic logic without theorems)

- $\mathcal{L}$ is **equivalential** when $\exists\, \Delta(x,y) \subseteq Fm$ satisfying (3a) , (3b) and (MP).

  (3b) $\quad \Delta(\alpha_1,\beta_1) \cup \cdots \cup \Delta(\alpha_n,\beta_n) \vdash_{\mathcal{L}} \Delta(\lambda\alpha_1 \ldots \alpha_n, \lambda\beta_1 \ldots \beta_n)$

  for each primitive connective λ, of arity n.

- $\mathcal{L}$ is **algebraizable** if and only if $\exists\, \Delta(x,y) \subseteq Fm$ and $\exists\, E(x) \subseteq Fm \times Fm$ such that conditions (3) , (MP) and (A3) are satisfied.

  (A3) $\quad \Delta\big(E(\alpha)\big) \dashv\vdash_{\mathcal{L}} \alpha$

- $\mathcal{L}$ is **regularly algebraizable** when $\exists\, \Delta(x,y) \subseteq Fm$ such that conditions (3) , (MP) and (5a) are satisfied.

  (5a) $\quad \alpha\,, \beta \vdash_{\mathcal{L}} \Delta(\alpha,\beta)$

Definability characterizations

- $\mathcal{L}$ is **equivalential** when $\exists\, \Delta(x,y) \subseteq Fm$ such that for every theory T of $\mathcal{L}$, $\boldsymbol{\Omega}_{\boldsymbol{Fm}} T$ is definable by $\Delta(x,y)$, i.e., for every $\alpha\,,\beta \in Fm$,

  (2) $$\alpha \equiv \beta \;\big(\boldsymbol{\Omega}_{\boldsymbol{Fm}} T\big) \iff \Delta(\alpha,\beta) \subseteq T.$$

  or, equivalently,

  $$\forall \langle \boldsymbol{A},F\rangle \in \mathbf{Mod}\mathcal{L}\,,\ \forall a,b \in A\,,\ \ a \equiv b\;\big(\boldsymbol{\Omega}_{\boldsymbol{A}} F\big) \iff \Delta^{\boldsymbol{A}}(a,b) \subseteq F.$$

- $\mathcal{L}$ is **protoalgebraic** when $\exists\, \Delta(x,y,\vec{z}) \subseteq Fm$ with parameters such that for every theory T of $\mathcal{L}$ and every $\alpha\,,\beta \in Fm$,

  (2') $$\alpha \equiv \beta \;\big(\boldsymbol{\Omega}_{\boldsymbol{Fm}} T\big) \iff \Delta(\alpha,\beta,\vec{\gamma}) \subseteq T \text{ for all } \vec{\gamma}.$$

  or, equivalently,

  $$\forall \langle \boldsymbol{A},F\rangle \in \mathbf{Mod}\mathcal{L}\,,\ \forall a,b \in A\,,\ \ a \equiv b\;\big(\boldsymbol{\Omega}_{\boldsymbol{A}} F\big) \iff \Delta^{\boldsymbol{A}}(a,b,\vec{c}) \subseteq F\ \ \forall \vec{c} \in A.$$

- $\mathcal{L}$ is **weakly algebraizable** when it is protoalgebraic and $\exists\, E(x) \subseteq Fm \times Fm$ such that the $\mathcal{L}$-filters in reduced models of $\mathcal{L}$ are definable by $E(x)$, i.e.,

  $$\text{if } \ \langle \boldsymbol{A},F\rangle \in \mathbf{Mod}^{*}\mathcal{L} \ \text{ then } \ F = \big\{a \in A : \boldsymbol{A} \models E(x)\,[\![a]\!]\big\}.$$

Lattice-theoretical characterizations

Conditions on $\boldsymbol{\Omega}_{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{L} \longrightarrow \mathrm{Co}_{\mathsf{Alg}^*\mathcal{L}}\boldsymbol{Fm}$,

or equivalently on $\boldsymbol{\Omega}_{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \longrightarrow \mathrm{Co}_{\mathsf{Alg}^*\mathcal{L}}\boldsymbol{A}$ for arbitrary $\boldsymbol{A}$

$\mathcal{L}$ is ...	iff	$\boldsymbol{\Omega}$ is ...
protoalgebraic		monotone
equivalential		monotone and commutes with inverse substitutions
finitely equivalential		monotone and continuous
weakly algebraizable		an isomorphism (monotone and injective)
algebraizable		an isomorphism and commutes with inverse substitutions
finitely algebraizable		a continuous isomorphism

Model-theoretic characterizations

$\mathcal{L}$ is protoalgebraic $\iff$ **Mod**$^*\mathcal{L}$ is closed under $\mathbb{P}_{SD}$.

$\mathcal{L}$ is equivalential $\iff$ **Mod**$^*\mathcal{L}$ is closed under $\mathbb{S}$ and $\mathbb{P}$.

$\mathcal{L}$ is finitely equivalential $\iff$ **Mod**$^*\mathcal{L}$ is closed under $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$, i.e., it is a quasivariety of matrices.

$\mathcal{L}$ is weakly algebraizable $\iff$ **Alg**$^*\mathcal{L}$ is closed under $\mathbb{P}_{SD}$ and in **Mod**$^*\mathcal{L}$ the filters are equationally definable.

$\mathcal{L}$ is algebraizable $\iff$ **Alg**$^*\mathcal{L}$ is closed under $\mathbb{S}$ and $\mathbb{P}$ and in **Mod**$^*\mathcal{L}$ the filters are equationally definable.

Classification of some protoalgebraic logics

Protoalgebraic but neither equivalential nor weakly algebraizable

- The logic in the language $\langle \to \rangle$ axiomatized by $\alpha \to \alpha$ and Modus Ponens.
- The logic in the language $\langle \to \rangle$ defined by the Gentzen calculus with all structural rules and:

$$\frac{\alpha\,,\beta \rhd \gamma}{\alpha \rhd \beta \to \gamma}\ (\text{DT1}) \qquad\qquad \frac{\Gamma \rhd \alpha \qquad \Gamma\,,\beta \rhd \gamma}{\Gamma\,,\alpha \to \beta \rhd \gamma}\ (\text{MP})$$

- DA COSTA's paraconsistent logic $\mathcal{C}1$.
- The logics defined from the *classical* modal systems E and RE with Modus Ponens as the only rule of inference.

Classification of some protoalgebraic logics

Equivalential but non-algebraizable

- The logics defined from the *normal* modal systems $\mathcal{K}$ and $\mathcal{T}$ with Modus Ponens as only rule; i.e., the local consequences defined by the classes of all Kripke frames and of all reflexive Kripke frames.

  (Not finitely equivalential)

- PRATT's dynamic logics. (Not finitely equivalential in general)

- The logics defined from the *normal* modal systems $\mathcal{S}4$ and $\mathcal{S}5$ with Modus Ponens as only rule; i.e., the local consequences defined by the classes of all reflexive and transitive Kripke frames and of all equivalence relations as Kripke frames.

- The implicational logics $\mathcal{BCI}$, $E_{\rightarrow}$ (implicational entailment) and $\mathcal{R}_{\rightarrow}$ (implicational relevance logic).

Classification of some protoalgebraic logics

Weakly algebraizable but non-equivalential

- The logic defined by $\{\langle \boldsymbol{A}, \{1\}\rangle : \boldsymbol{A} \text{ is an ortholattice}\}$.
  (Regularly)

- ERNST's logic, in $\langle \leftrightarrow \rangle$, axiomatized with $\alpha \leftrightarrow \alpha$, Modus Ponens for $\leftrightarrow$, and the infinite sets of rules:

$$\frac{\alpha}{\varphi(\alpha) \leftrightarrow \varphi(\alpha \leftrightarrow \alpha)} \quad \forall \varphi \qquad\qquad \frac{\alpha}{\varphi(\alpha \leftrightarrow \alpha) \leftrightarrow \varphi(\alpha)} \quad \forall \varphi$$

  (Not regularly)

Classification of some protoalgebraic logics

Algebraizable but not finitely algebraizable

- HERRMANN's logic LJ ("Last Judgement"), in the full modal language, axiomatized with Modus Ponens as only rule, and the axioms:
  - All instances of theorems of $\mathcal{CPL}$.
  - $\Box^n \varphi$ for all theorems of $\mathcal{IPL}$ and all $n \geqslant 0$.
  - $\Box^n \big(\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)\big)$ for all $n \geqslant 0$.
  - $(\alpha \to \beta) \to \Box^n(\neg\beta \to \neg\alpha)$ for all $n \geqslant 0$.

  (Not regularly)

- DELLUNDE's logic, in the language $\langle \Box , \leftrightarrow \rangle$, axiomatized by $\alpha \leftrightarrow \alpha$, Modus Ponens for $\leftrightarrow$, and all the rules:

$$\frac{\alpha \qquad \beta}{\Box^n\alpha \leftrightarrow \Box^n\beta} \quad \forall n \geqslant 0 \qquad\qquad \frac{\alpha_1 \leftrightarrow \beta_1 \qquad \alpha_2 \leftrightarrow \beta_2}{\Box^n(\alpha_1 \leftrightarrow \alpha_2) \leftrightarrow \Box^n(\beta_1 \leftrightarrow \beta_2)} \quad \forall n \geqslant 0$$

  (Regularly)

Classification of some protoalgebraic logics

Finitely algebraizable but not regularly

- Relevance logic $\mathcal{R}$.
- Some linear logics, some fuzzy logics.
- Substructural logics (Full Lambek calculus and extensions without weakening).

Regularly finitely algebraizable but not implicative

- $\mathcal{CPL}_{\leftrightarrow}$ and $\mathcal{IPL}_{\leftrightarrow}$, the equivalence fragments of $\mathcal{CPL}$ and of $\mathcal{IPL}$, resp.
- $\mathcal{IPL}_{\leftrightarrow,\neg}$, the fragment of $\mathcal{IPL}$ with equivalence and negation.

Some bridge theorems

For an arbitrary $\mathcal{L}$:

- $\mathcal{L}$ is finitary $\iff$ $\mathbf{Mod}\mathcal{L}$ is closed under ultraproducts.

For a protoalgebraic $\mathcal{L}$:

- $\mathcal{L}$ has the DDT $\iff$ $\mathbf{Mod}\mathcal{L}$ has definable principal $\mathcal{L}$-filters.
  $\iff$ $\mathbf{Mod}^*\mathcal{L}$ has definable principal $\mathcal{L}$-filters.
  $\iff$ For each A, the join-semilattice of the finitely generated $\mathcal{L}$-filters is dually Brouwerian.

- $\mathcal{L}$ has the LDT $\iff$ $\mathbf{Mod}\mathcal{L}$ has the "$\mathcal{L}$-filter extension property".
  $\iff$ $\mathbf{Mod}^*\mathcal{L}$ has the "$\mathcal{L}$-filter extension property".

For equivalential $\mathcal{L}$:

- $\mathcal{L}$ has Craig interpolation $\iff$ $\mathbf{Mod}\mathcal{L}$ has amalgamation.
  $\iff$ $\mathbf{Mod}^*\mathcal{L}$ has amalgamation.

Finally,

What about **non-protoalgebraic** logics ?

To be continued . . .