Abstract Algebraic Logic
An overview (I)

Josep Maria Font

Department of Probability, Logic and Statistics
University of Barcelona

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What is (abstract) algebraic logic ?

The study of the **connections** between **logics**
and **algebra-based semantics**.

- **Describe** these connections: Since BOOLE (1847)

  Completeness theorems and others

- **Exploit** these connections: Use the strength of algebra

  Bridge theorems

- **Explain** these connections: General theories

  General notion of a logic . . .

What is an "algebra-based semantics"?

- The structures taken as models are **algebras** $\boldsymbol{A} = \langle A, \tau^{\boldsymbol{A}} \rangle$ of the same similarity type τ as the language of the logic,

  sometimes endowed with some **additional structure**
  (**matrices, generalized matrices, ordered algebras, . . .**)

- **valuations** or **interpretations** are homomorphisms $v : \boldsymbol{Fm} \to \boldsymbol{A}$,

  $\forall\, \alpha \in Fm\,,\ v(\alpha) \in A$, and is computed from $\{v(x) : x \in Var(\alpha)\}$ using exclusively the algebraic structure,

  i.e., an algebra-based semantics is always a *truth-functional* semantics.

- **truth of α in the model** is defined by conditions involving $v(\alpha)$ and the algebraic structure, and perhaps the additional structure.

What is a logic ?

- Let τ be a sentential language, or algebraic similarity type.
- $\boldsymbol{Fm} = \langle Fm, \tau \rangle$ is the **formula algebra** or free term algebra of type τ. It is freely generated by some set Var of variables or atomic formulas.
- Our **logics** $\mathcal{L} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{L}} \rangle$ are **substitution-invariant consequence relations** over $\boldsymbol{Fm}$; i.e., relations $\vdash_{\mathcal{L}} \subseteq P(Fm) \times Fm$ such that:

  $\varphi \vdash_{\mathcal{L}} \varphi$.

  $\Gamma \vdash_{\mathcal{L}} \varphi$ implies $\Delta \vdash_{\mathcal{L}} \varphi$ whenever $\Gamma \subseteq \Delta$.

  $\Gamma \vdash_{\mathcal{L}} \varphi$ implies $\Delta \vdash_{\mathcal{L}} \varphi$ whenever $\Delta \vdash_{\mathcal{L}} \psi$ for each $\psi \in \Gamma$.

  $\Gamma \vdash_{\mathcal{L}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathcal{L}} \sigma(\varphi)$ for any substitution σ.
- The **theorems** of $\mathcal{L}$ are the $\alpha \in Fm$ such that $\emptyset \vdash_{\mathcal{L}} \alpha$.

- Logics not conceived as set of formulas, but as consequence relations.
- Independently of the way how $\vdash_{\mathcal{L}}$ is defined; i.e., using "$\vdash$" does not mean we assume any syntactical presentation!

An (algebraic) completeness of a logic $\mathcal{L}$ with respect to a class **K** of algebras

(1)
$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \forall \boldsymbol{A} \in \mathbf{K},\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}),\ \text{if } v(\gamma) = 1\ \forall \gamma \in \Gamma \text{ then } v(\varphi) = 1.$$

Proof. $(\Rightarrow)$: Routine checking, **K** and 1 wisely chosen.

$(\Leftarrow)$: Assume $\Gamma \nvdash_{\mathcal{L}} \varphi$, and **construct** some $\boldsymbol{A} \in \mathbf{K}$ and $v : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $v(\gamma) = 1\ \forall \gamma \in \Gamma$ while $v(\varphi) \neq 1$.

Heavily semantical-dependent constructions:

EXAMPLE: If one has another semantics for $\mathcal{L}$:

$\Gamma \nvdash_{\mathcal{L}} \varphi \Rightarrow \exists \mathfrak{M}, \bar{a}$ such that $\mathfrak{M} \models \gamma\ [\![\bar{a}]\!]\ \ \forall \gamma \in \Gamma$ but $\mathfrak{M} \not\models \varphi\ [\![\bar{a}]\!]$.

Then construct some $\boldsymbol{A} \in \mathbf{K}$ and $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that

$$\mathfrak{M} \models \alpha\ [\![\bar{a}]\!] \iff v(\alpha) = 1.$$

A syntactic construction: the LINDENBAUM-TARSKI process (the case of classical logic $\mathcal{CPL}$)

Let T be the $\mathcal{CPL}$**-theory generated by** Γ.

(2) *Define* $\alpha \equiv \beta\ (T) \iff \Gamma \vdash_{\mathcal{CPL}} \alpha \leftrightarrow \beta.$ $\quad (\iff \alpha \leftrightarrow \beta \in T)$

Show:

(3) $\equiv(T)$ is a **congruence** of the formula algebra $\boldsymbol{Fm}$.

(4) The quotient algebra $\boldsymbol{Fm}/{\equiv}(T) \in$ **BA**.

(5) $\alpha \in T \iff \alpha/{\equiv}(T) = 1$.

Take $\boldsymbol{A} := \boldsymbol{Fm}/{\equiv}(T)$ and $v(x) := x/{\equiv}(T) \quad \forall x \in Var$.

and then $v(\gamma) = 1 \ \ \forall \gamma \in \Gamma$ and $v(\varphi) \neq 1$.

Note: Condition **(5)** actually gathers three different facts:

(5a) $\alpha, \beta \in T \Rightarrow \alpha \equiv \beta\ (T)$.

(5b) $\alpha \in T,\ \alpha \equiv \beta\ (T) \Rightarrow \beta \in T$.

(5c) $T/{\equiv}(T) = 1$.

Some examples of work in “concrete” algebraic logic

Once you have the completeness theorem, you can exploit any kind of properties of your algebras in order to obtain better completeness theorems.

- Use **embedding/representation theorems** to get completeness wrt some special class $\mathbf{K} \subseteq \mathbf{BA}$.

  Find $\mathbf{K} \subseteq \mathbf{BA}$ such that $\forall \boldsymbol{A} \in \mathbf{BA}\ \exists \boldsymbol{B} \in \mathbf{K}$ and $\exists g : \boldsymbol{A} \to \boldsymbol{B}$ one-to-one.

  Then $g \circ v : \boldsymbol{Fm} \to \boldsymbol{B}$ also works, and you get completeness wrt $\mathbf{K}$.

- Since $\mathcal{CPL}$ is **finitary** and $\mathbf{BA}$ is **locally finite**, you get completeness wrt the class of finite Boolean algebras.

  Assume Γ is finite, and consider the subalgebra $\boldsymbol{B}$ of $\boldsymbol{Fm}/{\equiv}(T)$ generated by $\{v(x) : x \in Var(\Gamma, \varphi)\}$. Modify v outisde these so that it becomes $v : \boldsymbol{Fm} \to \boldsymbol{B}$.

- Using **ultrafilter properties** of Boolean algebras, you get completeness wrt the 2-element Boolean algebra $\mathbf{2}$.

  Since $v(\varphi) \neq 1$ there is an ultrafilter $\mathcal{U}$ of $\boldsymbol{Fm}/{\equiv}(T)$ such that $v(\varphi) \notin \mathcal{U}$. But $\big(\boldsymbol{Fm}/{\equiv}(T)\big)/\mathcal{U} \cong \mathbf{2}$, and $v(\varphi)/\mathcal{U} \neq 1$.

The first generalization of the process

- $\sim$ 1930 LINDENBAUM and TARSKI
- $\sim$ 1950 HENKIN; SIKORSKI and RASIOWA: sufficient conditions for logics to which the process can be applied without any changes
- RASIOWA [1974]: **implicative logics** $\boldsymbol{\mathcal{L}}$: Have a binary $\to$ such that:

(2): $\alpha \equiv \beta \; (T) \iff \alpha \to \beta \,, \beta \to \alpha \in T$

(3): $\equiv (T)$ is a congruence: $\vdash_{\mathcal{L}} \alpha \to \alpha \quad ; \quad \alpha \to \beta \,, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$

$$\left\{ \begin{matrix} \alpha_1 \to \beta_1 \,, \ldots, \alpha_n \to \beta_n \\ \beta_1 \to \alpha_1 \,, \ldots, \beta_n \to \alpha_n \end{matrix} \right\} \vdash_{\mathcal{L}} \lambda \alpha_1 \ldots \alpha_n \to \lambda \beta_1 \ldots \beta_n \quad \forall \lambda \in \tau$$

(5b): Modus Ponens $\quad \alpha \,, \alpha \to \beta \vdash_{\mathcal{L}} \beta$

(5a) and **(5c)**: Rule K $\quad \alpha \vdash_{\mathcal{L}} \beta \to \alpha$

(4): $\boldsymbol{A} \in$ **Alg**$^*\mathcal{L} \iff \exists 1 \in A$ such that part $(\Rightarrow)$ of **(1)** works

i.e., such that $\langle \boldsymbol{A} \,, \{1\} \rangle$ is a **model** of $\mathcal{L}$,

and $\boldsymbol{A} \models x \to y \approx 1 \;\&\; y \to x \approx 1 \Rightarrow x \approx y$.

Some classical examples

Classical logic	$\longleftrightarrow$	Boolean algebras
Intuitionistic logic	$\longleftrightarrow$	Heyting algebras
Positive implicative logic	$\longleftrightarrow$	Hilbert algebras
(Gödel-)Dummett's logic	$\longleftrightarrow$	"linear" Heyting algebras
Logic of constructive negation	$\longleftrightarrow$	Nelson's algebras
Normal modal logics (Global consequences)	$\longleftrightarrow$	Boolean algebras with operators (modal algebras, etc.)
Łukasiewicz's $Ł_\infty$	$\longleftrightarrow$	Wajsberg algebras (MV-algebras)
Post's many-valued logics	$\longleftrightarrow$	Post algebras
⋮	$\longleftrightarrow$	⋮

Inessential changes

Change the **truth condition** $v(\alpha) = 1$, i.e., the **truth set** $D = \{1\}$

- In many substructural logics (linear, fuzzy, relevance logic $\mathcal{RM}$):
  Replace "$v(\alpha) = 1$" by "$v(\alpha) \geqslant 1$" everywhere
  Take as truth set $D = \{a \in A : a \geqslant 1\}$
- Other relevance logics ($\mathcal{R}$):
  Replace "$v(\alpha) = 1$" by "$v(\alpha) \geqslant v(\alpha) \to v(\alpha)$" everywhere
  Take as truth set $D = \{a \in A : a \geqslant a \to a\}$

What is essential here ?

The **equational definability** of the truth condition "$v(\alpha) \in D$":

There is some set of equations $E(x) \subseteq Fm \times Fm$ such that

$$a \in D \iff \delta^{\boldsymbol{A}}(a) = \varepsilon^{\boldsymbol{A}}(a) \;\; \forall \delta \approx \varepsilon \in E(x) \iff \boldsymbol{A} \models E(x) \, [\![a]\!]$$

Algebraic semantics, the technical notion

Definition (Blok and Pigozzi, 1989)

A class **K** of algebras is an **algebraic semantics** for a logic $\mathcal{L}$

when there is a set of equations $E(x) \subseteq Fm \times Fm$ such that

(1') $\Gamma \vdash_{\mathcal{L}} \varphi \iff \forall \boldsymbol{A} \in \mathbf{K}$, $\forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$,

if $\boldsymbol{A} \models E(x)\ [\![v(\gamma)]\!]\ \ \forall \gamma \in \Gamma$ then $\boldsymbol{A} \models E(x)\ [\![v(\varphi)]\!]$.

(equations in $E(x)$ are called the **defining equations**)

$D = \{1\} \quad \longmapsto \quad E(x) = \{x \approx 1\}$ or $\{x \approx x \to x\}$

$D = \{a \in A : a \geqslant 1\} \quad \longmapsto \quad E(x) = \{x \vee 1 \approx x\}$ or $\{x \wedge 1 \approx 1\}$

$D = \{a \in A : a \geqslant a \to a\} \quad \longmapsto \quad E(x) = \{x \wedge (x \to x) \approx x \to x\}$

Algebraic semantics, the technical notion

The relative equational consequence $\vDash_{\mathsf{K}}$ intrinsically determined by a class of algebras K

$$\{\alpha_i \approx \beta_i : i \in I\} \vDash_{\mathsf{K}} \alpha \approx \beta \iff \forall \boldsymbol{A} \in \mathsf{K}, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}),$$

$$\text{if } \boldsymbol{A} \vDash \alpha_i \approx \beta_i \, [\![v]\!] \ \forall i \in I \text{ then } \boldsymbol{A} \vDash \alpha \approx \beta \, [\![v]\!].$$

$$\iff \mathsf{K} \vDash \mathop{\&}_{i \in I} \alpha_i \approx \beta_i \Rightarrow \alpha \approx \beta$$

Every set $E(x)$ determines a **translation** $\boldsymbol{\tau} : P(Fm) \to P(Fm \times Fm)$ by:

$$\boldsymbol{\tau}(\alpha) := E(\alpha) \qquad \forall \alpha \in Fm$$

$$\boldsymbol{\tau}(\Gamma) := \bigcup \{\boldsymbol{\tau}(\alpha) : \alpha \in \Gamma\} \qquad \forall \Gamma \subseteq Fm$$

(1') amounts to: $\boldsymbol{\Gamma} \vdash_{\mathcal{L}} \varphi \iff \boldsymbol{\tau}(\boldsymbol{\Gamma}) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$

($\boldsymbol{\tau}$ is a **faithful interpretation** of $\vdash_{\mathcal{L}}$ into $\vDash_{\mathsf{K}}$)

The paradigm of the algebraization of a logic ?

- Having an algebraic semantics is a rather weak property.
- Not every logic has an algebraic semantics.
- An algebraic semantics for a logic can be rather weird.
- The algebraic semantics for a logic need not be unique,

  e.g., for $\mathcal{CPL}$ we have:
  - **BA**
  - { finite Boolean algebras }
  - $\{\mathbf{2}\}$

  but also:
  - { dually complemented bounded distributive lattices }
  - { Heyting algebras } with $E(x) = \{\neg\neg x \approx 1\}$ (GLIVENKO's theorem)

- How to characterize the “right” one ?

What is essential in the properties of $\equiv (T)$?

(2) There is some set $\Delta(x,y) \subseteq Fm$ such that $\alpha \equiv \beta \; (T) \iff \Delta(\alpha,\beta) \subseteq T$ satisfies **(3)** and **(5b)**.

- Examples

$\Delta(x,y) = \{x \leftrightarrow y\}$ — $\mathcal{CPL}$, $\mathcal{IPL}$, *global* normal modal, many-valued, etc.

$\Delta(x,y) = \{x \to y \, , y \to x\}$ — $\mathcal{IPL}_{\to}$, etc.

$\Delta(x,y) = \{x \leftrightarrow y \, , \Box(x \leftrightarrow y) \, , \Box^2(x \leftrightarrow y) \, , \dots\}$ — *local* modal $\mathcal{K}$, $\mathcal{T}$

$\Delta(x,y) = \{\Box(x \leftrightarrow y)\}$ — *local* modal $\mathcal{S}4$, $\mathcal{S}5$

$\Delta(x,y) = \{x \to y \, , y \to x \, , \neg x \to \neg y \, , \neg y \to \neg x\}$ — $\mathcal{P}1$, $Ł_3$

What is essential in the properties of $\equiv(T)$?

(2) There is some set $\Delta(x,y) \subseteq Fm$ such that $\alpha \equiv \beta\ (T) \iff \Delta(\alpha,\beta) \subseteq T$ satisfies **(3)** and **(5b)**.

- Whenever such a $\Delta(x,y)$ exists such that $\equiv(T)$ satisfies **(3)** and **(5b)**, then it is the largest relation satisfying them. [PORTE's Theorem]
- The formulas in $\Delta(x,y)$ are called **equivalence formulas**.
- Every set $\Delta(x,y)$ determines a **translation** $\boldsymbol{\rho} : P(Fm \times Fm) \to P(Fm)$ by:

$$\boldsymbol{\rho}(\alpha \approx \beta) := \Delta(\alpha,\beta) \qquad \forall \alpha \approx \beta \in Fm \times Fm$$

$$\boldsymbol{\rho}(\Theta) := \bigcup\{\boldsymbol{\rho}(\alpha \approx \beta) : \alpha \approx \beta \in \Theta\} \qquad \forall \Theta \subseteq Fm \times Fm$$

and for $\mathbf{K} = \mathbf{Alg}^*\mathcal{L}$, $\boldsymbol{\rho}$ is a **faithful interpretation** of $\models_{\mathbf{K}}$ into $\vdash_{\mathcal{L}}$,

i.e., the *dual* of **(1')** holds:

$$\boldsymbol{\Theta \models_{\mathbf{K}} \delta \approx \varepsilon \iff \rho(\Theta) \vdash_{\mathcal{L}} \rho(\delta \approx \varepsilon)}$$

The notion of an algebraizable logic

Definition (Blok and Pigozzi, 1989)

A logic $\mathcal{L}$ is **algebraizable** when there exists a class of algebras **K** and structural translations τ, ρ which are **mutually inverse faithful interpretations** of $\vdash_{\mathcal{L}}$ into $\models_{\mathbf{K}}$ and conversely, i.e., such that:

(A1) $\Gamma \vdash_{\mathcal{L}} \varphi \iff \tau(\Gamma) \models_{\mathbf{K}} \tau(\varphi)$

(A2) $\Theta \models_{\mathbf{K}} \delta \approx \varepsilon \iff \rho(\Theta) \vdash_{\mathcal{L}} \rho(\delta \approx \varepsilon)$

(A3) $\varphi \dashv\vdash_{\mathcal{L}} \rho\big(\tau(\varphi)\big)$

(A4) $\delta \approx \varepsilon \mathrel{=}\!\mid\models_{\mathbf{K}} \tau\big(\rho(\delta \approx \varepsilon)\big)$

The class **K** is called an **equivalent algebraic semantics** of $\mathcal{L}$

Actually, (A1) + (A4) $\iff$ (A2) + (A3)

- An equivalent algebraic semantics need not be unique.
- There is always the largest equivalent algebraic semantics: **the algebraic counterpart** of $\mathcal{L}$.
- For implicative logics, it coincides with $\mathbf{Alg}^*\mathcal{L}$.
- If $\mathcal{L}$ is finitary and $\boldsymbol{\tau}, \boldsymbol{\rho}$ are finite then it coincides with $\mathbb{Q}(\mathbf{K})$. It is the only equivalent quasivariety.
- All implicative logics are algebraizable.
- The other logics treatable with "inessential changes", are algebraizable.
- If $\mathcal{L}$ is algebraizable wrt $\mathbf{K}$ then any fragment of $\mathcal{L}$ having the connectives appearing in $E(x)$ and in $\Delta(x, y)$ is algebraizable, wrt $\{$subreducts of $\mathbf{K}\}$.
- If $\mathcal{L}$ is algebraizable wrt $\mathbf{K}$, then any extension of $\mathcal{L}$ is algebraizable, wrt a subclass of $\mathbf{K}$, and using the same $\boldsymbol{\tau}, \boldsymbol{\rho}$.
- If a finitary $\mathcal{L}$ is algebraizable wrt $\mathbf{K}$ with finite $\boldsymbol{\tau}, \boldsymbol{\rho}$, then any extension of $\mathcal{L}$ is algebraizable, wrt a subquasivariety of $\mathbf{K}$.
- If moreover $\mathcal{L}$ is algebraizable wrt a variety $\mathbf{K}$ then the axiomatic extensions of $\mathcal{L}$ are algebraizable and correspond to the subvarieties of $\mathbf{K}$.
- These correspondences are lattice isomorphisms.

Kinds of algebraizability

- Not exactly BLOK and PIGOZZI's 1989 original definition.
- Extended by HERRMANN, CZELAKOWSKI, FONT and JANSANA.

finitely algebraizable: The translation ρ (i.e., the set Δ) is finite.

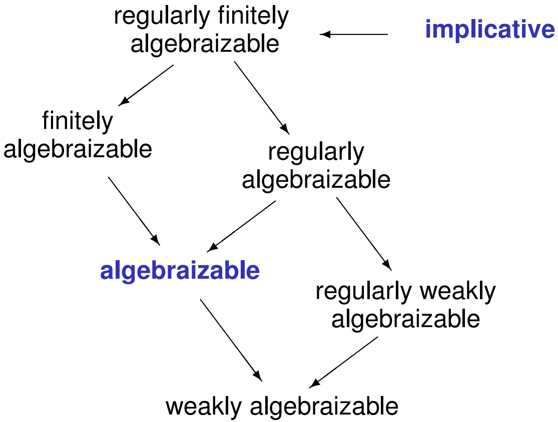
"BP-algebraizable": Finitary and finitely algebraizable.
Then also τ (i.e., the set E) is finite.

regularly algebraizable: The truth set is unitary. Equivalently, condition **(5a)** holds, i.e., $\alpha, \beta \vdash_{\mathcal{L}} \Delta(\alpha, \beta)$.

strongly algebraizable: The largest equivalent algebraic semantics $\mathbf{Alg}^*\mathcal{L}$ is a variety.

weakly algebraizable: Similar, weaker notion, where the equivalence formulas can contain parameters.
Few proper examples; theoretical interest.

(a part of) the algebraizable hierarchy



A syntactic intrinsic characterization

Theorem

A logic $\mathcal{L}$ is ***algebraizable***

if and only if

there are formulas $\Delta(x,y) \subseteq Fm$ and equations $E(x) \subseteq Fm \times Fm$ such that:

(3a) $\vdash_{\mathcal{L}} \Delta(\alpha,\alpha)$

(3b) $\Delta(\alpha_1,\beta_1) \cup \cdots \cup \Delta(\alpha_n,\beta_n) \vdash_{\mathcal{L}} \Delta(\lambda\alpha_1 \ldots \alpha_n, \lambda\beta_1 \ldots \beta_n)$
for each primitive connective λ, of arity n

(5b) $\alpha, \Delta(\alpha,\beta) \vdash_{\mathcal{L}} \beta$

(A3) $\Delta\big(E(\alpha)\big) \dashv\vdash_{\mathcal{L}} \alpha$

Note: (3a) + (3b) + (5b) $\Longrightarrow$
- (3c) $\Delta(\alpha,\beta) \vdash_{\mathcal{L}} \Delta(\beta,\alpha)$
- (3d) $\Delta(\alpha,\beta) \cup \Delta(\beta,\gamma) \vdash_{\mathcal{L}} \Delta(\alpha,\gamma)$

Exploiting algebraizability: "bridge" theorems

If $\mathcal{L}$ is strongly "BP-algebraizable", then:

$\mathcal{L}$ has property $P \iff$ **K** has property P'

Examples

finite axiomatizability	$\longleftrightarrow$	finite presentation
decidability	$\longleftrightarrow$	decidability
the deduction theorem	$\longleftrightarrow$	having EDPC
the local deduction theorem	$\longleftrightarrow$	congruence extension property
Craig's interpolation theorem	$\longleftrightarrow$	amalgamation
Beth's definability theorem	$\longleftrightarrow$	epimorphisms are surjective
regularity rule (5a)	$\longleftrightarrow$	congruence-regularity

The most famous bridge theorem

Theorem (Blok and Pigozzi, actually ca. 1978 !)

Let $\mathcal{L}$ be a strongly BP-algebraizable logic, with equivalent variety **K***.*

Then $\mathcal{L}$ has the Deduction-Detachment Theorem

if and only if

K *has equationally definable principal congruences (EDPC).*

Facts

If a variety **K** has EDPC then:

- **K** is congruence-distributive.
- **K** has the congruence-extension property.
- $\{\boldsymbol{A} \in \mathbf{K} : \boldsymbol{A}$ is simple$\}$ is closed under ultraproducts.
- $\{\boldsymbol{A} \in \mathbf{K} : \boldsymbol{A}$ is subdirectly irreducible$\}$ is closed under ultraproducts.

These facts allow to disprove that a BP-algebraizable $\mathcal{L}$ has the DDT.

Importance of the notion of algebraizability

- It is a precise, mathematical formulation of a vague, informal idea.
- It has originated a rich and deep mathematical theory.
- It can be equivalently characterized from different points of view:
  - Syntactic (properties of $\vdash_{\mathcal{L}}$ using Δ and E).
  - Semantic (properties of **K**, of the congruences of algebras in **K**, etc.)
  - Lattice-theoretic (properties of the mapping $T \mapsto \equiv(T)$ on the theories of $\mathcal{L}$)
- Allows to show that a given logic $\boldsymbol{\mathcal{L}}$ is not algebraizable (wrt any class of algebras and any conceivable translations).
- Allows to show that a given class **K** of algebras is not "logifiable" (it cannot be the equivalent algebraic semantics of any algebraizable logic).

Finally,

What about **non-algebraizable** logics ?

To be continued . . .