On the infinite-valued ŁUKASIEWICZ logic that preserves degrees of truth

Josep Maria Font
University of Barcelona

Àngel J. Gil
University Pompeu Fabra

Antoni Torrens
University of Barcelona

Ventura Verdú
University of Barcelona

Submitted version, July 6, 2005

Abstract

ŁUKASIEWICZ's infinite-valued logic is commonly defined as the set of formulas that take the value 1 under all evaluations in the ŁUKASIEWICZ algebra on the unit real interval. In the literature a deductive system axiomatized in a Hilbert style was associated to it, and was later shown to be semantically defined from ŁUKASIEWICZ algebra by using a "truth-preserving" scheme. This deductive system is algebraizable, non-selfextensional and does not satisfy the deduction theorem. In addition, there exists no Gentzen calculus fully adequate for it. Another presentation of the same deductive system can be obtained from a substructural Gentzen calculus. In this paper we use the framework of abstract algebraic logic to study a different deductive system which uses the aforementioned algebra under a scheme of "preservation of degrees of truth". We characterize the resulting deductive system in a natural way by using the lattice filters of Wajsberg algebras, and also by using a structural Gentzen calculus, which is shown to be fully adequate for it. This logic is an interesting example for the general theory: it is selfextensional, non-protoalgebraic, and satisfies a "graded" deduction theorem. Moreover, the Gentzen system is algebraizable. The first mentioned deductive system turns out to be the extension of the second by the rule of Modus Ponens.

MSC 2000 Classification: 03B50, 03G20, 06D35, 03B22.

Key words and phrases: Many-valued logic, truth degrees, Wajsberg algebras, MV-algebras, Gentzen systems, algebraic logic, non-protoalgebraic logic, selfextensional logic, algebraizable logic.

1 Introduction

By the infinite-valued Łukasiewicz logic one commonly refers to the set of formulas in a suitable language[1] $\mathcal{L}$ that hold in the so-called Łukasiewicz algebra over the real unit interval $[\mathbf{0,1}]$, when taking as *designated* the only element 1. Łukasiewicz himself conjectured a finite axiomatization of this set of *tautologies*. The conjecture was proved right by Wajsberg (lost) and later on by Rose and Rosser [30], and independently by Chang [5]. Concerning *consequence*, it was proved by Hay [21] that the finitary deductive system $L_\infty = \langle \mathcal{L}, \vdash_\infty \rangle$ axiomatized by those tautologies and the rule of Modus Ponens coincides with that axiomatized by Łukasiewicz's axioms and Modus Ponens, and also with the finitary logic semantically determined by the associated matrix, with 1 as the only designated element; see [6, 20, 34] for properties of this deductive system and more references. As to its abstract algebraic logic status, Rodríguez, Torrens and Verdú proved in [29] that L_∞ is finitely, strongly and regularly algebraizable, that its equivalent algebraic semantics is the class W of all Wajsberg algebras (better known as MV-algebras, in a polinomially equivalent presentation), and that on each $\boldsymbol{A} \in \mathsf{W}$ the L_∞-filters coincide with the *implicative filters*, i.e., those $F \subseteq A$ such that $1 \in F$ and F is closed under Modus Ponens (if $a, a \to b \in F$ then $b \in F$). As a consequence, L_∞ is complete with respect to the class of matrices

(1) $\big\{ \langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{W}\,, F \subseteq A \text{ is an implicative filter} \big\}.$

The logic L_∞ also satisfies the following completeness result: For all *finite* $\Gamma \subseteq Fm$ and all $\varphi \in Fm$,

(2) $$\Gamma \vdash_\infty \varphi \iff \text{For all } v \in \operatorname{Hom}(\boldsymbol{Fm}, [\mathbf{0,1}]) : \text{ if } v(\beta) = 1 \text{ for all } \beta \in \Gamma, \text{ then } v(\varphi) = 1,$$

where $\boldsymbol{Fm}$ is the algebra of $\mathcal{L}$-formulas. If one takes 1 as representing "truth" in the algebra of truth-values $[\mathbf{0,1}]$, then one can interpret this result as saying that L_∞ is the *truth-preserving* logic associated with the Łukasiewicz algebra $[\mathbf{0,1}]$. However, as Wójcicki points out in [34, p. 279], "there is no evidence that Łukasiewicz thought of the consequence operations characteristic of many-valued logics as truth-preserving ones".

A different approach has more recently come from the field of substructural logics. There Łukasiewicz's logic appears as an extension of one of the basic substructural Gentzen systems, and the algebraic structures are placed in the framework of residuated lattices; see [25] for an introductory survey. The basic intuitions under this approach are mainly proof-theoretical, and still the corresponding deductive system is L_∞, but the link between the Gentzen system and the deductive system is not the standard one, and its sequents do not represent entailment in the deductive system.

In this paper we explore a third alternative, namely that of defining a different deductive system $L_\infty^{\leqslant}$ by using the ordering relation of the algebra $[\mathbf{0,1}]$ under a scheme of *preservation of degrees of truth*, and studying it within the framework of abstract algebraic logic [13]. In particular we will show that $L_\infty^{\leqslant}$ admits a structural Gentzen system with a unique natural relation with the

[1] See Section 2 for technical details.

logic (namely, the Gentzen system is fully adequate for the logic, as is proved in Corollary 4.3) and with a smooth algebraic treatment.

The idea of preservation of degrees of truth we use in this paper, as opposed to that of truth-preservation, considers all elements of the truth-value algebra **[0,1]** as genuine truth values without privileging one among them. Thus, it interprets *consequence* in the following sense: That whenever all premisses attain at least a certain degree of truth, the conclusion should attain at least that degree of truth too. To implement this idea one only needs to use the ordering relation $\leqslant$ that exists among the elements of **[0,1]**, but needs not make any assumption as to the meaning of connectives. Thus the deductive system preserving degrees of truth associated with this structure is given by:

$$(3) \quad \Gamma \vdash_{\infty}^{\leqslant} \varphi \iff \text{For all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \mathbf{[0,1]}) \text{ and all } t \in [0,1] : \text{ if } v(\beta) \geqslant t \text{ for all } \beta \in \Gamma, \text{ then } v(\varphi) \geqslant t.$$

where Γ is any *finite* set of formulas, and φ is any formula.

Incidentally, note that schemes (2) and (3), when taken for an arbitrary Γ, are known to define consequence relations on the formula algebra which are not finitary ([9]), thus they fall outside our theoretical framework, where a deductive system is always assumed to be finitary.

We do believe that this definition is natural enough to give the idea of "degrees of truth" a sound interpretation, and that in this way one can avoid the puzzling difficulties encountered by Dana Scott in his well-known and challenging discussion of many-valued logics ([31, 32]), and in particular his scepticism about the role of designated truth values in a really many-valued logic.

An early use of a definition similar to (3) is found in Cleave's analysis ([7]) of Kleene's 3-valued logic. For more general discussions on schemes of preservation of degrees of truth and their application to many-valued logics, see [9, 24, 26]. The study of particular finite- and infinite-valued Łukasiewicz logics defined by this scheme was initiated by Wójcicki in [34, Section 4.3][2] with his definition of the logics $\mathcal{L}_{\xi}^{(\leqslant\models}$, and was followed by Gil, Torrens and Verdú in [16, 17] and by Font in [9]. The publication [17] is an abstract without proofs, and is not easily accessible, hence we include and prove in the present paper some of its main results.

There is a striking difference between the results of applying either of the two previously discussed schemes when defining more general *infinite-valued* Łukasiewicz *logics*. Since their early appearance in [23] this denomination has been used to refer to any of the logics defined by the scheme of preservation of truth as in (2) but applied to an infinite subalgebra of **[0,1]**. It is well-known [9, 19] that in this way we obtain an infinite number of distinct logics. By contrast, it was proved in [9, Theorem 22] that if we apply a scheme of preservation of degrees of truth as in (3) to any infinite subalgebra of **[0,1]** we always obtain the same logic, the one denoted by $\mathcal{L}_{\aleph_0}^{(\leqslant\models}$ by Wójcicki and which is studied in this paper. Thus speaking about *the* infinite-valued Łukasiewicz logic that preserves degrees of truth appears to be fully justified.

[2] Warning: In this book the terms "truth-preserving" and "preserving degrees of truth" are used in the senses explained above, except on page 345; at this place "truth-preserving" means basically the same as the former "preserving degrees of truth", and the term "validity-preserving" is used to mean the same as the former "truth-preserving".

From our results it turns out that $L_\infty^{\leqslant} = \langle \mathcal{L}, \vdash_\infty^{\leqslant} \rangle$ is very interesting both from the point of view of the general theory of Abstract Algebraic Logic and for its specifically many-valued properties:

- It is *non-protoalgebraic* (Theorem 3.11). This means that neither its implication connective, nor any other definable binary connective, can satisfy both the laws of *Identity* and *Modus Ponens*.
- It satisfies a simple *graded deduction theorem* [12] (Lemma 3.2).
- It can be defined from the class of all Wajsberg algebras if we take all *lattice filters* as designated sets (compare Definition 3.1 and Lemma 3.2).
- It is *selfextensional*, i.e., the interderivability relation in $L_\infty^{\leqslant}$ is a congruence relation on the algebra of formulas (Theorem 3.4). Since it also has a conjunction, some general results in [11] can be applied to it. For instance, this property automatically ensures that the algebraic counterpart of the logic is a variety, and indeed we prove that it is the class of Wajsberg algebras (Corollary 3.5).
- Another consequence of selfextensionality is that there is a structural Gentzen system that is fully adequate for the logic and algebraizable. One of the aims of this paper is to present an explicit, finite presentation of this system (Theorem 4.10).

The paper is organized as follows: in Section 2 we recall some basic notions of algebraic logic and some properties of the Łukasiewicz logic L_∞ and of Wajsberg algebras. In Section 3 we define the logic $L_\infty^{\leqslant}$, and prove its basic algebraic properties, and those relating it to L_∞; in particular we show that the two logics have the same theorems and that L_∞ is an extension of $L_\infty^{\leqslant}$ that can be obtained by adding a single inference rule to $L_\infty^{\leqslant}$. We also classify $L_\infty^{\leqslant}$ according to several general criteria of abstract algebraic logic: we prove that it is selfextensional and it is not protoalgebraic, hence, it is not algebrizable either. In Section 4 we introduce a finitely presented sequent calculus, and prove that the models of its associated Gentzen system $\mathfrak{G}_\infty$ are essentially the full generalized models of $L_\infty^{\leqslant}$, which means that $\mathfrak{G}_\infty$ is fully adequate for $L_\infty^{\leqslant}$ in the sense of [11, 13]. Using this property we obtain several ways of describing the relation between $\mathfrak{G}_\infty$ and $L_\infty^{\leqslant}$, and a characterization of $L_\infty^{\leqslant}$ in terms of abstract properties of its consequence relation; moreover we show that the *external deductive system* (in the sense of [1]) associated with $\mathfrak{G}_\infty$ is L_∞. Finally, in Section 5 we further study the models of $\mathfrak{G}_\infty$ and use them to characterize the full generalized models of L_∞ in several ways. The actual derivations in a Gentzen system needed for the proofs of some lemmas in sections 4 and 5 have been placed in the Appendix.

Acknowledgements. The authors were partially supported by grants MTM2004-03101 and TIN2004-07933-C03-02 of the Spanish Ministry of Education and Science, including FEDER funds of the European Union, and by grant 2001SGR-00017 of the Catalan Department of Universities, Research and Information Society.

2 Some background

Given an algebraic language $\mathcal{L}$, let us denote its associated formula algebra by $\boldsymbol{Fm}_{\mathcal{L}}$ or simply by $\boldsymbol{Fm}$. In this paper we define a (propositional) ***logic*** or ***deductive system*** $\mathcal{S}$ as a pair $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}}$ is a finitary consequence relation over $Fm_{\mathcal{L}}$ which is invariant under substitutions. The algebraic models we are going to consider are different kinds of structures over algebras of the similarity type of the logic, i.e., $\mathcal{L}$-algebras $\boldsymbol{A}$; and the valuations will be just homomorphisms from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$.

The simpler models are, of course, just algebras. But algebras provide rich enough models only for the best-behaved logics, among which our $L_{\infty}^{\leqslant}$ will not be, in some sense. Thus we need more complex models, ***logical matrices***, which are pairs $\langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an algebra and $F \subseteq A$ is the ***filter***, or designated subset of the matrix. Such a matrix is a ***model*** of a logic $\mathcal{S}$ when, for all $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $v(\gamma) \in F$ for all $\gamma \in \Gamma$ then $v(\varphi) \in F$. The set F is then called an ***$\mathcal{S}$-filter***, and $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ denotes the set of all $\mathcal{S}$-filters over $\boldsymbol{A}$. Since this set is a closed-set system, it has an associated closure operator: For any $X \subseteq A$, $Fi_{\mathcal{S}}^{\boldsymbol{A}}(X)$ is the least $\mathcal{S}$-filter on $\boldsymbol{A}$ containing X. The third kind of models are ***generalized matrices*** (g-matrices for short), which in this paper will be pairs $\langle \boldsymbol{A}, \mathcal{C} \rangle$ where $\mathcal{C} \subseteq P(A)$ is an inductive closed-set system of subsets of A, i.e., a family containing A and closed under intersections of arbitrary subfamilies and under unions of upwards-directed subfamilies. The associated closure operator is denoted as C. A g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a ***generalized model*** (***g-model*** for short) of $\mathcal{S}$ when $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; equivalently, when for all $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ then $v(\varphi) \in C\big(\{v(\gamma) : \gamma \in \Gamma\}\big)$. For more details on these basic notions see [8, 13, 34].

A special kind of g-models has been introduced and exploited in [11] and a most general theory has been built around them. Using the most recent terminology we say that a ***basic full g-model*** of $\mathcal{S}$ is one of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \rangle$, for an arbitrary algebra $\boldsymbol{A}$, and a ***full g-model*** of $\mathcal{S}$ is a g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ such that there is a strict surjective homomorphism (also called bilogical mosphism) from it onto a basic full g-model of the logic. This kind of g-models constitutes the main tool on which the general theory of [11] is built. The consideration of reduced models leads to the definition of several classes of algebras associated with a logic. A matrix $\langle \boldsymbol{A}, F \rangle$ is ***reduced*** when the only congruence of $\boldsymbol{A}$ that is compatible with the filter F (i.e., a $\theta \in \mathrm{Co}\boldsymbol{A}$ such that if $a\theta b$ and $a \in F$ then also $b \in F$) is the identity; and a g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is ***reduced*** when the identity is the only congruence of $\boldsymbol{A}$ that is compatible with all the filters in $\mathcal{C}$. Then three classes of algebras can be associated in general to an arbitrary deductive system:

$$\begin{aligned}
\mathsf{Alg}^{*}\mathcal{S} &= \{\boldsymbol{A} : \exists\, F \subseteq S \text{ such that } \langle \boldsymbol{A}, F \rangle \text{ is reduced}\} \\
\mathsf{Alg}\mathcal{S} &= \{\boldsymbol{A} : \exists\, \mathcal{C} \subseteq P(A) \text{ such that } \langle \boldsymbol{A}, \mathcal{C} \rangle \text{ is a reduced g-matrix}\} \\
\mathbf{V}(\mathcal{S}) &= \text{ the variety generated by any of the previous classes.}
\end{aligned}$$

(That the two first classes generate the same variety, and the same quasi-variety, is proved in Proposition 2.26 of [11].) For protoalgebraic logics $\mathsf{Alg}^{*}\mathcal{S} = \mathsf{Alg}\mathcal{S}$ and there is no doubt that this is the algebraic counterpart of the logic. However for non-protoalgebraic logics this equality may not hold, and it seems that in general it is $\mathsf{Alg}\mathcal{S}$ the class that allows for a smoother expression of the links between the logic and the algebras. For algebraizable logics this class coincides with the so-called equivalent algebraic semantics for $\mathcal{S}$, see [3, 8, 13].

From now on we fix a language $\mathcal{L} = \langle \to, \neg \rangle$ of type $(2,1)$ and denote by $\boldsymbol{Fm}$ the corresponding formula algebra. We will omit mention of the particular similarity type unless it is necessary. We consider also the defined connectives $\vee$, $\wedge$, $*$ and $\oplus$ given by:

$$
\begin{aligned}
x \vee y &= (x \to y) \to y, \\
x \wedge y &= \neg(\neg x \vee \neg y), \\
x * y &= \neg(x \to \neg y), \\
x \oplus y &= \neg x \to y.
\end{aligned} \tag{4}
$$

We also define x^n as $x^{n-1} * x$ for $n \geqslant 2$ and x^1 as x. The connectives $\vee$ and $\wedge$ will be the usual lattice-like operations of disjunction and conjunction, resp. The operation $*$ is called ***fusion*** or multiplicative conjunction (in the literature it is also denoted as $\cdot$ or $\odot$) and $\oplus$ is the multiplicative disjunction.

The ***(infinite-valued)* Łukasiewicz *logic*** $Ł_\infty = \langle \mathcal{L}, \vdash_\infty \rangle$ is the logic defined over $\boldsymbol{Fm}$ with axioms:

(A1) $\varphi \to (\psi \to \varphi)$

(A2) $(\varphi \to \psi) \to \big((\psi \to \xi) \to (\varphi \to \xi)\big)$

(A3) $\big((\varphi \to \psi) \to \psi\big) \to \big((\psi \to \varphi) \to \varphi\big)$

(A4) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

and with *Modus Ponens* as the only primitive rule: $\varphi\,, \varphi \to \psi \vdash \psi$. The corresponding algebras are the so called ***Wajsberg algebras***, i.e., algebras $\boldsymbol{A} = \langle A, \to, \neg \rangle$ of type $(2,1)$ that satisfy the following equations:

(W1) $(x \to x) \to y \approx y$.

(W2) $(x \to y) \to \big((y \to z) \to (x \to z)\big) \approx x \to x$.

(W3) $(x \to y) \to y \approx (y \to x) \to x$.

(W4) $(\neg x \to \neg y) \to (y \to x) \approx x \to x$.

These algebras are polynomially equivalent to Wajsberg algebras as originally defined by Rodríguez in [28], and to Chang's MV-algebras defined in [5]: equations (4) give one half of the definitions, and for the converse use $x \to y = \neg x \oplus y$. The main general references for Wajsberg algebras and MV-algebras are [6, 14, 20]. The variety of all Wajsberg algebras, denoted by W, is generated by the algebra $\mathbf{[0,1]}$ on the unit real interval, with the well known Łukasiewicz operations $\neg x = 1 - x$ and $x \to y = \min\{1, 1 - x + y\}$ (where $+$ and $-$ are the ordinary arithmetical operations). The operations induced on $\mathbf{[0,1]}$ by (4) are $x \vee y = \max\{x,y\}\,, x \wedge y = \min\{x,y\}\,, x * y = \max\{0, x + y - 1\}$ and $x \oplus y = \min\{1, x+y\}$. The finite simple algebras in this variety are (up to isomorphism) the finite subalgebras of $\mathbf{[0,1]}$: for each $m \geqslant 2$ we denote by $\boldsymbol{S_m}$ the m-element algebra with universe $\{0, 1/(m-1), \ldots, (m-2)/(m-1), 1\}$. These algebras define finite-valued logics in a similar way as $Ł_\infty^{\leqslant}$ is defined in (3); they have been studied in [9, 16, 18].

If $\boldsymbol{A}$ is a Wajsberg algebra then $\boldsymbol{A} \models x \to x \approx y \to y$, hence these terms define an algebraic constant, denoted by 1; we also put $0 = \neg 1$. If for any $a, b \in A$ we define $a \leqslant b \iff a \to b = 1$, then $\leqslant$ is a partial order relation, which endows $\boldsymbol{A}$ with a bounded distributive lattice structure with the join given by $\vee$, the meet given by $\wedge$ and with maximum 1 and minimum 0. This lattice satisfies De Morgan laws (hence, it is a De Morgan algebra [2]) and it is residuated in the sense that for any $a, b, c \in A$,

(5) $$a * b \leqslant c \iff a \leqslant b \to c$$

The following identities that hold in W will be used in the paper:

(6) $$x * (x \to y) \approx x \wedge y$$

(7) $$x \to (y \to z) \approx y \to (x \to z)$$

(8) $$(x \wedge y)^2 \preccurlyeq x * y$$

(9) $$x^{n+1} \preccurlyeq x^n$$

We use the symbol $\approx$ to denote formal equations, as is usual. Similarly, we denote by $\varphi \preccurlyeq \psi$ the "formal ordering" relation that holds in an algebra $\boldsymbol{A}$ endowed with a partial order $\leqslant$ when for all $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, $v(\varphi) \leqslant v(\psi)$. Note that if the algebra is a lattice or a semilattice (like those in W) then $\varphi \preccurlyeq \psi$ is actually equivalent to an equation, namely to the equation $\varphi \wedge \psi \approx \varphi$.

The main relationships between Wajsberg algebras and Łukasiewicz logic are summarized in the following theorem. Its part (a) establishes the relationship between the infinite-valued Łukasiewicz logic and the ***implicative filters*** of Wajsberg algebras, i.e., the sets F such that $1 \in F$ and are closed under Modus Ponens, i.e., such that if $a, a \to b \in F$ then $b \in F$; the set of all implicative filters of an algebra $\boldsymbol{A}$ will be denoted by $\mathcal{F}_{\to}(\boldsymbol{A})$ or simply by $\mathcal{F}_{\to}$, and the implicative filter generated by X will be denoted by $F^{\boldsymbol{A}}_{\to}(X)$ or by $F_{\to}(X)$. Parts (b)–(e) follow Blok and Pigozzi's theory of algebraizable logics [3, 4] and Font and Jansana's theory of full g-models [11].

Theorem 2.1 ([14, 29])

(a) *The logic* $Ł_\infty$ *is complete with respect to the class of matrices*

$$\big\{ \langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{W} \text{ and } F \text{ is an implicative filter on } \boldsymbol{A} \big\}.$$

That is, for any $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_\infty \varphi \iff \text{For all } v \in \mathrm{Hom}(\boldsymbol{Fm}, [\mathbf{0,1}]) \text{ and all } F \in \mathcal{F}_{\to}(\boldsymbol{A}) : \text{ if } v(\beta) \in F \text{ for all } \beta \in \Gamma, \text{ then } v(\varphi) \in F.$$

(b) $Ł_\infty$ *is an algebraizable logic. Its equivalence formulas are* $\{\varphi \to \psi, \psi \to \varphi\}$, *its defining equation is* $\varphi \approx 1$, *and its largest equivalent algebraic semantics is the variety* W.

(c) $\mathsf{Alg}(Ł_\infty) = \mathsf{W}$.

(d) *The reduced models of* $Ł_\infty$ *are all matrices* $\langle \boldsymbol{A}, \{1\} \rangle$ *with* $\boldsymbol{A} \in \mathsf{W}$.

(e) *The reduced full g-models of* L_∞ *are all g-matrices of the form* $\langle \boldsymbol{A}, \mathcal{F}_{\to}(\boldsymbol{A})\rangle$ *with* $\boldsymbol{A} \in \mathbf{W}$.

Actually, the properties included in part (b) tell us not just that L_∞ is algebraizable, but that it is also, using the modern terminology, *finitely algebraizable* (the number of equivalence formulas and of defining equations is finite), *strongly algebraizable* (the associated class of algebras is a variety) and *regularly algebraizable* (the reduced matrices have one-element filters).

The role played by the implicative filters of a Wajsberg algebra in the previous result will be played by ***lattice filters*** for the logic we are going to introduce in the next section. If $\boldsymbol{A}$ is a Wajsberg algebra, the set of all its lattice filters will be denoted by $\mathcal{F}_{\leqslant}(\boldsymbol{A})$, or simply by $\mathcal{F}_{\leqslant}$, and the lattice filter generated by a subset $X \subseteq A$, that is, the smallest lattice filter in which D is contained, denoted as $F_{\leqslant}(X)$ or $F_{\leqslant}^{\boldsymbol{A}}(X)$, can be characterized as

(10) $$F_{\leqslant}(X) = \{b \in A : \text{ there exist } a_1, \ldots, a_n \in X \text{ with } b \geqslant a_1 \wedge \cdots \wedge a_n\}.$$

Every implicative filter is a lattice filter, and a lattice filter is an implicative filter if and only if it is closed under Modus Ponens. In Lemma 3.8 we give other conditions for a lattice filter to be an implicative filter. It will be useful to record the characterization of the implicative filter generated by a set X and an element a in a Wajsberg algebra, as given in [6, Proposition 4.2.9] or [33, 1.19]:

(11) $$F_{\to}(X, a) = \{b \in A : \text{ there exists } n \geqslant 1 \text{ such that } a^n \to b \in F_{\to}(X)\}.$$

This property is the algebraic expression of the so-called *local deduction theorem* for L_∞, see [13, Section 3.2].

3 The logic

We first define the logic we are going to study and later on (Corollary 3.3) we prove that it is indeed tha logic that preserves degrees of truth in $\mathbf{[0,1]}$, according to (3):

Definition 3.1 *The logic* $L_\infty^{\leqslant} = \langle \mathcal{L}, \vdash_\infty^{\leqslant}\rangle$ *is the logic defined by the class of matrices* $\big\{\langle \boldsymbol{A}, H\rangle : \boldsymbol{A} \in \mathbf{W}\ ,\ H$ *is a lattice filter on* $\boldsymbol{A}\big\}$*. That is, for any* $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_\infty^{\leqslant} \varphi$ *iff for every* $\boldsymbol{A} \in \mathbf{W}$*, every* $H \in \mathcal{F}_{\leqslant}(\boldsymbol{A})$ *and every* $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$*,* $v[\Gamma] \subseteq H$ *implies* $v(\varphi) \in H$*.*

Since we are using *all* lattice filters on each Wajsberg algebra, the definition can be equivalently formulated using the lattice filter-generation operator:

(12) $$\Gamma \vdash_\infty^{\leqslant} \varphi \iff \forall \boldsymbol{A} \in \mathbf{W}\ ,\ \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ ,\ v(\varphi) \in F_{\leqslant}\big(v[\Gamma]\big).$$

In particular, for $\Gamma = \emptyset$ we see that φ is a theorem of $L_\infty^{\leqslant}$ iff its interpretations in all Wajsberg algebras always belong to the least lattice filter, which is the set $\{1\}$. Hence we conclude that

(13) $$\emptyset \vdash_\infty^{\leqslant} \varphi \iff \mathbf{W} \models \varphi \approx 1.$$

The following lemma contains the first elementary properties of $L_\infty^{\leqslant}$. Its last item concerns the ***interdervability relation***, which for an arbitrary logic $\mathcal{S}$ is defined as

(14) $$\varphi \dashv\vdash_{\mathcal{S}} \psi \iff \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi.$$

This relation is always an equivalence relation. When it is a congruence of the formula algebra, the logic will be called ***selfextensional*** [34]; these logics enjoy a good replacement property:

$$\text{If } \varphi \dashv\vdash_{\mathcal{S}} \psi \text{ then } \delta(\varphi) \dashv\vdash_{\mathcal{S}} \delta(\psi)$$

for any formula δ with at least one propositional variable. It is known that L_∞ is not selfextensional [9, Proposition 13], for instance $p \dashv\vdash_{L_\infty} p * p$ but $\neg p \not\dashv\vdash_{L_\infty} \neg(p * p)$.

We say that a logic $\langle \mathcal{L}, \vdash' \rangle$ is an ***extension*** of the logic $\langle \mathcal{L}, \vdash \rangle$ when $\vdash \subseteq \vdash'$, and that the extension is ***proper*** when $\vdash \subsetneq \vdash'$. An extension $\langle \mathcal{L}, \vdash' \rangle$ is called ***axiomatic*** when it can be obtained by adding only axioms to $\langle \mathcal{L}, \vdash \rangle$, and it is called ***purely inferential*** when it can be obtained by adding only proper rules to $\langle \mathcal{L}, \vdash \rangle$, but has the same theorems.

Lemma 3.2

(a) *$L^{\leqslant}_\infty$ is a finitary deductive system.*

(b) *L_∞ is a proper, purely inferential extension of $L^{\leqslant}_\infty$. The theorems of both logics are the formulas φ such that $\mathbf{W} \models \varphi \approx 1$, i.e., such that $[\mathbf{0,1}] \models \varphi \approx 1$.*

(c) *For all $\varphi_1, \dots, \varphi_n, \psi \in Fm$ the following are equivalent:*

  (i) $\varphi_1, \dots, \varphi_n \vdash^{\leqslant}_\infty \psi$.

  (ii) $\mathbf{W} \models \varphi_1 \wedge \dots \wedge \varphi_n \preccurlyeq \psi$.

  (iii) $[\mathbf{0,1}] \models \varphi_1 \wedge \dots \wedge \varphi_n \preccurlyeq \psi$.

  (iv) $\emptyset \vdash^{\leqslant}_\infty \varphi_1 \wedge \dots \wedge \varphi_n \to \psi$, *i.e.,* $\emptyset \vdash_\infty \varphi_1 \wedge \dots \wedge \varphi_n \to \psi$.

(d) *For all $\varphi, \psi \in Fm$, $\varphi \dashv\vdash^{\leqslant}_\infty \psi \iff \mathbf{W} \models \varphi \approx \psi$.*

Proof

(a) It follows from the fact that $\mathbf{W}$ is a variety and the notion of lattice filter is first-order definable, so the class of matrices that defines $L^{\leqslant}_\infty$ is closed under the formation of ultraproducts.

(b) Since an implicative filter in a Wajsberg algebra is also a lattice filter, from (1) and Definition 3.1 it follows that $\Gamma \vdash^{\leqslant}_\infty \varphi$ implies $\Gamma \vdash_\infty \varphi$, that is, L_∞ is an extension of $L^{\leqslant}_\infty$. From Theorem 2.1(b) it follows that $\emptyset \vdash_\infty \varphi \iff \mathbf{W} \models \varphi \approx 1$, hence by (13) the theorems of the two logics coincide. Finally, it is clear that $L^{\leqslant}_\infty$ does not satisfy Modus Ponens.

(c) The equivalence between (i) and (ii) follows from the application of (10) to (12). As we have already pointed out, in $\mathbf{W}$ the formal expression $\varphi \preccurlyeq \psi$ is equivalent to an equation (namely, to $\varphi \wedge \psi \approx \varphi$), hence (ii) is equivalent to (iii). Finally, we also have

$$\mathbf{W} \models \varphi_1 \wedge \dots \wedge \varphi_n \preccurlyeq \psi \iff \varphi_1 \wedge \dots \wedge \varphi_n \to \psi \approx 1,$$

and this together with part (b) shows the equivalence between (ii) and (iv).

(d) This is an immediate consequence of (c). ∎

Corollary 3.3 *$L_\infty^\leqslant$ is the logic that preserves degrees of truth in* $\mathbf{[0,1]}$*, that is, for all finite $\Gamma \subseteq Fm$ and all $\varphi \in Fm$,*

$$(15) \quad \Gamma \vdash_\infty^\leqslant \varphi \iff \text{For all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \mathbf{[0,1]}) \text{ and all } t \in [0,1] : \\ \text{if } v(\beta) \geqslant t \text{ for all } \beta \in \Gamma, \quad \text{then } v(\varphi) \geqslant t.$$

Proof The result follows from the equivalence between (i) and (iii) in part (c) of Lemma 3.2 and the fact that, in a lattice, $t \leqslant v(\varphi_i)$ for $i = 1, \dots, n$ iff $t \leqslant v(\varphi_1) \wedge \cdots \wedge v(\varphi_n)$. ∎

In addition, the equivalence between (i) and (iv), more precisely, the property that

$$\varphi_1, \dots, \varphi_n \vdash_\infty^\leqslant \psi \iff \emptyset \vdash_\infty^\leqslant \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi$$

is an instance of what has been called in [12] a *graded deduction theorem* for $L_\infty^\leqslant$. In this case it is a particularly remarkable property because, as we show in Theorem 3.11, $L_\infty^\leqslant$ is not a protoalgebraic logic, hence there is no hope that neither $\to$ nor any other defined implication satisfies the ordinary deduction theorem; moreover up to now that property had been investigated only in the context of protoalgebraic logics.

The first elements of the classification of $L_\infty^\leqslant$ in the hierarchies of abstract algebraic logic are the following:

Theorem 3.4 *$L_\infty^\leqslant$ is selfextensional and satisfies the property of conjunction, i.e., the connective $\wedge$ satisfies the rules $x, y \vdash x \wedge y$, $x \wedge y \vdash x$, $x \wedge y \vdash y$.*

Proof Selfextensionality follows from Lemma 3.2(d). The three rules that define the property of conjunction are also straightforward. ∎

Thus we can apply to this logic the general results obtained in [11] for selfextensional logics with conjunction and having theorems. The first is the confirmation that Wajsberg algebras are the right algebraic counterpart of this logic:

Corollary 3.5 $\mathsf{Alg}^*(L_\infty^\leqslant) = \mathsf{Alg}(L_\infty^\leqslant) = \mathsf{V}(L_\infty^\leqslant) = \mathsf{W}$.

Proof From Lemma 4.27 of [11] it follows that for a selfextensional logic $\mathcal{S}$ with conjunction the classes of algebras $\mathsf{Alg}^*\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ and $\mathsf{V}(\mathcal{S})$ coincide; and by Lemma 2.43 of [11] this variety is defined by the equations $\varphi \approx \psi$ such that $\varphi \dashv\vdash_\mathcal{S} \psi$. Then part (d) of our Lemma 3.2 ends the proof. ∎

Corollary 3.6 *If $\boldsymbol{A} \in \mathsf{W}$, then $F \in \mathcal{F}i_{L_\infty^\leqslant}\boldsymbol{A}$ iff F is a lattice filter on $\boldsymbol{A}$.*

Proof From the property of conjunction it follows that every $L_\infty^\leqslant$-filter is a lattice filter. The converse follows from the very definition of $L_\infty^\leqslant$. ∎

By Proposition 2.17 of [11], the reduced full g-models of a logic coincide with its reduced basic full g-models, and these are the g-matrices of the form $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$ with $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. Now the two preceding corollaries yield:

Corollary 3.7 *The reduced full generalized models of $L_\infty^\leqslant$ are the generalized matrices of the form $\langle \boldsymbol{A}, \mathcal{F}_\leqslant(\boldsymbol{A})\rangle$, with $\boldsymbol{A}$ a Wajsberg algebra.* ∎

To further analyze the relationship between L_∞ and $L_\infty^\leqslant$ the following algebraic lemma concerning implicative filters in Wajsberg algebras will be useful.

Lemma 3.8 *Let $\boldsymbol{A} \in \mathsf{W}$ and let $F \in \mathcal{F}_\leqslant(\boldsymbol{A})$. Then the following properties are equivalent:*

(i) *F is an implicative filter.*

(ii) *F is closed under $*$, i.e., if $a, b \in F$ then $a * b \in F$.*

(iii) *F is square-closed, i.e., $a \in F$ implies $a^2 \in F$.*

(iv) *F is n-th power-closed, i.e., $a \in F$ implies $a^n \in F$ for all $n \geqslant 2$.*

Proof The equivalence between (i) and (ii) follows from [6, 4.7.2]. Either of (ii) and (iv) trivially implies (iii), and by using (9) it is easy to see that (iii) implies (iv). Finally, (iii) implies (ii) follows from (8). ∎

In Corollary 3.2 we have proved that L_∞ is a purely inferential extension of $L_\infty^\leqslant$. Now, by comparing Theorem 2.1(a) with Definition 3.1, Lemma 3.8 yields the following:

Corollary 3.9 *L_∞ is the extension of $L_\infty^\leqslant$ by any of the following rules or set of rules:*

(a) $p, p \to q \vdash q$.

(b) $p, q \vdash p * q$.

(c) $p \vdash p^2$.

(d) $p \vdash p^n$ *for all* $n \geqslant 2$. ∎

Continuing with the classification of logics in abstract algebraic logic we have the following results:

Theorem 3.10 *The logic $L_\infty^\leqslant$ is not algebraizable.*

Proof Consider the three-element Wajsberg algebra $\boldsymbol{S_3}$, which has the set $\{0, 1/2, 1\}$ as universe, and consider the lattice filters $\{1/2, 1\}$ and $\{1\}$. If is not difficult to check that $\Omega_{\boldsymbol{S_3}}(\{1/2, 1\}) = \Omega_{\boldsymbol{S_3}}(\{1\}) = Id$, therefore the Leibniz operator is not injective on the $L_\infty^\leqslant$-filters, and according to Theorem 5.1 of [3] the logic $L_\infty^\leqslant$ cannot be algebraizable. ∎

In the proof of the next result the following abbreviation is used: $1\,a = a$ and for $n \geqslant 2$, $n\,a = (n-1)\,a \oplus a$.

THEOREM 3.11 *The logic* $L_\infty^{\leqslant}$ *is not protoalgebraic.*

PROOF Let $\boldsymbol{A}$ be the direct product of the algebras $\boldsymbol{S_m}$ for all $m \geqslant 2$ (see [6, Chapter 6] or [33, Section 3.A]), and take $a = \langle 1/n : n \in \omega \rangle \in \boldsymbol{A}$. Let $F = F_{\rightarrow}(\neg a)$ be the implicative filter generated by $\neg a$; this set is both an $L_\infty^{\leqslant}$-filter (i.e., a lattice filter) and an L_∞-filter. By Theorem 2.1(b), the Leibniz congruences of its filters are defined by the equivalence formulas $\{p \rightarrow q\,, q \rightarrow p\}$. Since $a \rightarrow 0 = \neg a \in F$ by construction, and $0 \rightarrow a = 1 \in F$ because all filters contain the maximum, we have that $\langle a, 0\rangle \in \boldsymbol{\Omega_A}(F)$. If by contradiction we assume that $L_\infty^{\leqslant}$ is protoalgebraic, then $\boldsymbol{\Omega_A}$ should be monotonic on $L_\infty^{\leqslant}$-filters, and hence $\langle a, 0\rangle \in \boldsymbol{\Omega_A}(F')$ where $F' = F_{\leqslant}(F\,, a)$ is the lattice filter generated by $F \cup \{a\}$, which is also an $L_\infty^{\leqslant}$-filter. Since $a \in F'$ by compatibility we should also have $0 \in F' = F_{\leqslant}(F\,, a)$. Since F is itself a lattice filter, by standard lattice-filter generation this means that $0 = b \wedge a$ for some $b \in F = F_{\rightarrow}(\neg a)$. Using the characterization (11) of implicative filters in Wajsberg algebras, for $X = \emptyset$, plus the fact that $F_{\rightarrow}(\emptyset) = \{1\}$ we obtain that $(\neg a)^n \leqslant b$ for some $n \geqslant 1$. Therefore $(\neg a)^n \wedge a = 0$, which, by the definition of a turns out to be false. Therefore $L_\infty^{\leqslant}$ cannot be protoalgebraic. ∎

4 The Gentzen system

Let $\mathcal{L}$ be a propositional language. For the purposes of this paper, a ***sequent*** is a pair $\langle \Gamma, \varphi\rangle$ were Γ is a finite subset, possibly empty, of Fm and φ is a single formula. Since the symbols $\rightarrow, \Rightarrow, \vdash$ are used with other meanings, we will denote the sequent $\langle \Gamma, \varphi\rangle$ by $\Gamma \rhd \varphi$. We denote by $Seq_{\mathcal{L}}$ the set of $\mathcal{L}$-sequents. A ***Gentzen system*** is like a deductive system but with sequents instead of formulas. That is, it is a pair $\mathfrak{G} = \langle \mathcal{L}, \mid\!\sim_{\mathfrak{G}}\rangle$ where $\mid\!\sim_{\mathfrak{G}}$ is a finitary consequence relation on the set of sequents of the language $\mathcal{L}$ that is also substitution-invariant in the obvious sense, where a substitution σ applies to a sequent in the following way: $\sigma(\Gamma \rhd \varphi) = \sigma[\Gamma] \rhd \sigma(\varphi)$. Usually sequent systems will be defined by specifying a Gentzen-style *calculus*, that is, by giving some initial sequents and some rules or rule schemes, and defining a proof (or derivation) in the usual, Hilbert-style way, but with sequents instead of formulas. Here we are not interested only in the derivable sequents (those that are provable from the initial sequents) but in the consequence relation that arises when allowing proofs from a set of sequents taken as assumptions. By finitarity, the consequence is completely described by the *derivable rules* of the system. Thus, when we write

$$\{\Gamma_i \rhd \varphi_i : i < n\} \mid\!\sim_{\mathfrak{G}} \Gamma \rhd \varphi$$

this is the same as saying that the rule

$$\frac{\{\Gamma_i \rhd \varphi_i : i < n\}}{\Gamma \rhd \varphi}$$

is a derived rule of the Gentzen system $\mathfrak{G}$.

The sequent systems we want to consider will satisfy all the structural rules. For this reason we have taken sets of formulas in the left-hand side of sequents, rather than finite sequences or multisets; in this way we can dispense with the exchange and contraction rules. Also because of this reason, the natural objects to take as algebraic models of our sequent systems are generalized matrices.

Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be a g-matrix, with associated closure operator C, and let $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. We say that v ***satisfies a sequent*** $\Gamma \rhd \varphi$ when $v(\varphi) \in C\big(v[\Gamma]\big)$. From this definition the usual derived notions follow: A g-matrix is a ***model of a sequent*** when all its valuations satisfy it, and it is a ***model of a Gentzen system*** $\mathfrak{G}$ when for any set of sequents $\{\Gamma_i \rhd \varphi_i : i \in I\}$ and any sequent $\Gamma \rhd \varphi$ such that $\{\Gamma_i \rhd \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma \rhd \varphi$, any $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ that satisfies all $\Gamma_i \rhd \varphi_i$ also satifies $\Gamma \rhd \varphi$. For a finite I we also get the notion of ***model of a rule***. This notion of model corresponds, *mutatis mutandis*, to the notion of a matrix being a model of a deductive system. Accordingly, the whole classical theory of matrices and its ramifications can also be developped for g-matrices as models of Gentzen systems, and one naturally finds ***the algebraic counterpart of*** $\mathfrak{G}$, the class of algebras $\mathbf{Alg}\mathfrak{G}$ made of the algebraic reducts of its reduced models. The notion of algebraizability can be extended to a Gentzen system in terms of equivalence between the consequence of the Gentzen system and the relative equational consequence of a quasi-variety or a variety of algebras, when this equivalence is expressed by suitable translations. For algebraizable Gentzen systems, the largest equivalent algebraic semantics coincides with $\mathbf{Alg}\mathfrak{G}$. See [11, 27] and also [13, Section 4.2] for a more detailed discussion. For finitely-valued logics of the most general kind, a parallel study using m-sided sequents instead of the ordinary (two-sided, single-conclusion) ones has been developed in [16, 18].

The idea underlying our use of sequent systems is that the sequents represent the entailments of the deductive system under study. More precisely, given a logic $\mathcal{S}$ with theorems[3] and a Gentzen system $\mathfrak{G}$ we say that $\mathfrak{G}$ is ***adequate*** for $\mathcal{S}$ when $\mathcal{S}$ is the logic defined by the derivable sequents of $\mathfrak{G}$, that is, when for any $\Gamma \cup \{\varphi\} \subseteq Fm$

$$\Gamma \vdash_{\mathcal{S}} \varphi \iff \text{there is a finite subset } \Gamma_0 \subseteq \Gamma \text{ such that } \emptyset \mathrel{|\!\sim}_{\mathfrak{G}} \Gamma \rhd \varphi.$$

Every deductive system has an adequate Gentzen system, and it can have more than one. One of the tasks undertaken in [11] is to single out, among all these, one that bears a special relationship to the logic, and is moreover unique in this respect. To this end the generalized matrices were used in their double role as generalized models of deductive systems and as models of Gentzen systems. $\mathfrak{G}$ is said to be ***fully adequate*** for $\mathcal{S}$ when the class of all full g-models of $\mathcal{S}$ concides with the class of models of $\mathfrak{G}$; in other words, when the full g-models of $\mathcal{S}$ can be described by a set of Gentzen-style rules; this makes easier to describe the class $\mathbf{Alg}\mathcal{S}$ and has other nice consequences. It is easy to see that a fully adequate Gentzen system for a logic is also adequate. The property of having a fully adequate Gentzen system appears to be related to a number of quite different issues in abstract algebraic logic, as shown in [11, 12]; see [10, 13] for survey-like expositions. Actually not every deductive system has a fully adequate Gentzen system:

[3] Certain notions of the general theory have to be defined differently for logics without theorems, but since the logics we study do have them, we make this assumption from now on.

Theorem 4.1 *The logic L_∞ does not have a fully adequate Gentzen system.*

Proof Corollary 5.7 of [12] shows that a weakly algebraizable logic with a fully adequate Gentzen system satisfies the deduction theorem. All algebraizable logics are *a fortiori* weakly algebraizable, hence we can apply this result to L_∞. Since it is well known that L_∞ does not satisfy the deduction theorem (see [8, Thm. 2.6.9]), we conclude that L_∞ does not have a fully adequate Gentzen system. ∎

By contrast, selfextensional logics with conjunction do have a fully adequate Gentzen system, although the general theory does not guarantee that it admits a finite presentation. Let us summarize the main definitions and results in [11] concerning this issue:

Definition 4.2 *Let $\mathcal{S}$ be a deductive system. The Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is defined by the following initial sequents and rule schemes:*

(a) *The proper initial sequents $\Gamma \rhd \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$.*

(b) *The "structural rules", that is:*

(ax) *The initial sequent $\varphi \rhd \varphi$ for every $\varphi \in Fm$.*

(w) *The rule* $\dfrac{\Gamma \rhd \varphi}{\Gamma, \psi \rhd \varphi}$ *for every $\Gamma \subseteq Fm, \varphi, \psi \in Fm$.*

(cut) *The rule* $\dfrac{\Gamma \rhd \varphi \qquad \Gamma, \varphi \rhd \psi}{\Gamma \rhd \psi}$ *for every $\Gamma \subseteq Fm, \varphi, \psi \in Fm$.*

(c) *The "congruence rules"*

(cong) $\dfrac{\{\, \varphi_i \rhd \psi_i \,, \psi_i \rhd \varphi_1 : i = 1, \ldots, n \,\}}{\varpi\varphi_1, \ldots, \varphi_n \rhd \varpi\psi_1, \ldots, \psi_n}$

for each connective $\varpi \in \mathcal{L}$, where n is its arity.

Theorem 4.3 *Let $\mathcal{S}$ be a selfextensional logic with the property of conjunction. Then:*

(a) *The Gentzen system $\mathfrak{G}_{\mathcal{S}}$ is adequate and fully adequate for $\mathcal{S}$.*

(b) *$\mathfrak{G}_{\mathcal{S}}$ is algebraizable in the sense of* [27, 18].

(c) $\mathbf{Alg}\mathfrak{G}_{\mathcal{S}} = \mathbf{Alg}\mathcal{S}$ *and this class coincides with the variety* $\mathbf{V}(\mathcal{S})$. ∎

By Theorem 3.4 the logic $L_\infty^\leqslant$ satisfies the assumptions in this result, therefore there is a Gentzen system $\mathfrak{G}_{L_\infty^\leqslant}$ that is fully adequate for $L_\infty^\leqslant$, is algebraizable and such that its algebraic counterpart is $\mathbf{Alg}\mathfrak{G}_{L_\infty^\leqslant} = \mathbf{W}$. However, Definition 4.2 does not give a finite presentation of $\mathfrak{G}_{L_\infty^\leqslant}$, because it takes as initial sequents *all* sequents corresponding to finite entailments of the logic $L_\infty^\leqslant$. In the rest of this section we give a finite presentation of $\mathfrak{G}_{L_\infty^\leqslant}$, which is a slightly modified version of the one presented in [17].

Definition 4.4 *The Gentzen system* $\mathfrak{G}_\infty$ *is the one given by the following initial sequents and rule schemes; recall that* $\wedge, \vee$ *and* $*$ *are defined in* (4):

(ax) $\varphi \rhd \varphi$

(w) $\dfrac{\Gamma \rhd \varphi}{\Gamma , \psi \rhd \varphi}$ (cut) $\dfrac{\Gamma \rhd \varphi \qquad \Gamma , \varphi \rhd \psi}{\Gamma \rhd \psi}$

(id$\to$) $\rhd \varphi \to \varphi$ (ex$\to$) $\varphi \to (\psi \to \xi) \rhd \psi \to (\varphi \to \xi)$

($\wedge\rhd$) $\dfrac{\Gamma , \varphi \rhd \xi}{\Gamma , \varphi \wedge \psi \rhd \xi} \, , \, \dfrac{\Gamma , \psi \rhd \xi}{\Gamma , \varphi \wedge \psi \rhd \xi}$ ($\rhd\wedge$) $\dfrac{\Gamma \rhd \varphi \quad \Gamma \rhd \psi}{\Gamma \rhd \varphi \wedge \psi}$

($\vee\rhd$) $\dfrac{\Gamma , \varphi \rhd \xi \qquad \Gamma , \psi \rhd \xi}{\Gamma , \varphi \vee \psi \rhd \xi}$ ($\rhd\vee$) $\dfrac{\Gamma \rhd \varphi}{\Gamma \rhd \varphi \vee \psi} \, , \, \dfrac{\Gamma \rhd \psi}{\Gamma \rhd \varphi \vee \psi}$

($\neg$) $\dfrac{\varphi \rhd \psi}{\neg\psi \rhd \neg\varphi}$

($\neg\neg\rhd$) $\dfrac{\Gamma , \varphi \rhd \xi}{\Gamma , \neg\neg\varphi \rhd \xi}$ ($\rhd\neg\neg$) $\dfrac{\Gamma \rhd \varphi}{\Gamma \rhd \neg\neg\varphi}$

(res1) $\dfrac{\varphi * \psi \rhd \xi}{\varphi \rhd \psi \to \xi}$ (res2) $\dfrac{\varphi \rhd \psi \to \xi}{\varphi * \psi \rhd \xi}$

The labels "(id$\to$)*" and "*(ex$\to$)*" account for "identity" and "exchange" for the implication connective, respectively, while "*(res1)*" and "*(res2)*" refer to the "residuation property" of implication with respect to fusion.*

Notice that, as we have already observed, it is not necessary to put the structural rules of exchange and contraction explicitly, because the left-hand side of our sequents are finite *sets* of formulas. Thus this Gentzen system satisfies all structural rules. In some proofs it may be useful to recall that the rules ($\wedge\rhd$) can be replaced here, modulo the structural rules, by the following one:

($\wedge$) $\dfrac{\Gamma , \varphi , \psi \rhd \xi}{\Gamma , \varphi \wedge \psi \rhd \xi}$

Now we begin to analyze its models, and their relationship with the g-models of $\L_\infty^{\leqslant}$.

Lemma 4.5 *If* $\boldsymbol{A} \in \mathsf{W}$*, then the g-matrix* $\langle \boldsymbol{A}, \mathcal{F}_\leqslant(\boldsymbol{A}) \rangle$ *is a model for* $\mathfrak{G}_\infty$*.*

Proof Note that in every Wajsberg algebra, the g-matrix $\langle \boldsymbol{A}, \mathcal{F}_\leqslant(\boldsymbol{A}) \rangle$ satisfies a sequent of the form $\varphi_1, \dots, \varphi_n \rhd \psi$ for a valuation $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ iff $v(\varphi_1) \wedge \cdots \wedge v(\varphi_n) \leqslant v(\psi)$, and a sequent of the form $\emptyset \rhd \varphi$ iff $v(\varphi) = 1$. Now, each statement that $\langle \boldsymbol{A}, \mathcal{F}_\leqslant(\boldsymbol{A}) \rangle$ is a model of one of the initial sequents and rules of $\mathfrak{G}_\infty$ amounts to a straightforward property of the order structure of any Wajsberg algebra. For instance, (ax) corresponds to $x \preccurlyeq x$, (ex$\to$) corresponds to (7), (res1) and (res2) correspond to (5), and so on. ∎

COROLLARY 4.6 *The full g-models of* $L_\infty^{\leqslant}$ *are models of* $\mathfrak{G}_\infty$.

PROOF Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a full g-model of $L_\infty^{\leqslant}$; by Corollary 3.7 its reduction is of the form $\langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant}(\boldsymbol{A}^*)\rangle$ with $\boldsymbol{A}^* \in \mathbf{W}$. By 4.5 we know that $\langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant}(\boldsymbol{A}^*)\rangle$ is a model of $\mathfrak{G}_\infty$. Now, the reduction mapping from $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ is a strict epimorphism (bilogical morphism) and by Proposition 2.5 of [11] the property of being a model of a Gentzen-styule rule is preserved under taking images and inverse images by strict epimorphisms. Therefore, the g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a model of $\mathfrak{G}_\infty$ as well. ∎

Now we state several properties of the Gentzen system $\mathfrak{G}_\infty$ that will be used in the sequel. We will use the following notational abbreviation: For any formulas φ, ψ, we write $\varphi \lhd\rhd \psi$ to denote the set of two sequents $\{\varphi \rhd \psi, \psi \rhd \varphi\}$, so that when writing $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \lhd\rhd \psi$ we mean that each of the two sequents is derivable.

LEMMA 4.7 *In the Gentzen system* $\mathfrak{G}_\infty$ *the following properties hold:*

(16) $\varphi \rhd \psi \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \xi \to \varphi \rhd \xi \to \psi$.

(17) $\varphi \rhd \psi \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi * \xi \rhd \psi * \xi$.

(18) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \psi \to \neg\varphi \rhd \varphi \to \neg\psi$.

(19) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} (\varphi \to \neg\psi) \to \xi \rhd (\psi \to \neg\varphi) \to \xi$.

(20) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi * \psi \rhd \psi * \varphi$.

(21) $\varphi \rhd \psi \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \psi \to \xi \rhd \varphi \to \xi$.

(22) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \to \psi \lhd\rhd \neg\psi \to \neg\varphi$

(23) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \wedge \psi \lhd\rhd \psi \wedge \varphi$.

(24) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} (\varphi \to \psi) * \varphi \lhd\rhd \varphi \wedge \psi$.

(25) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} (\varphi \to \varphi) * \psi \lhd\rhd \psi$.

(26) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \rhd \varphi \vee \psi$,
$\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \rhd \psi \vee \varphi$, *and*
$\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \vee \psi \lhd\rhd \psi \vee \varphi$.

PROOF See Appendix 6. ∎

The following properties of the set of derivable sequents of the Gentzen system $\mathfrak{G}_\infty$ will be used to prove that $\mathfrak{G}_{L_\infty^{\leqslant}}$ and $\mathfrak{G}_\infty$ are equal.

LEMMA 4.8 *In the Gentzen system* $\mathfrak{G}_\infty$ *the following properties hold:*

(a) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi \to (\varphi \to \varphi)$.

(b) *If* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi$ *and* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \psi$ *then* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi * \psi$.

(c) $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi \to \psi$ *if and only if* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \varphi \rhd \psi$.

(d) *If* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi$ *and* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi \to \psi$ *then* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \psi$.

(e) *If* φ *is an axiom of* L_∞, *then* $\emptyset \mathrel{|\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi$.

PROOF See Appendix 6. ∎

Lemma 4.9 *Let $\Gamma \cup \{\gamma_1, \ldots, \gamma_n, \varphi\} \subseteq Fm$, where Γ is a finite set. Then*

(a) *If φ is a theorem of L_∞ then $\emptyset \mid\!\sim_{\mathfrak{G}_\infty} \emptyset \rhd \varphi$.*

(b) *If $\gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$ is a theorem of L_∞ then $\emptyset \mid\!\sim_{\mathfrak{G}_\infty} \gamma_1, \ldots, \gamma_n \rhd \varphi$.*

(c) *If $\Gamma \vdash^{\leqslant}_{\infty} \varphi$ then $\emptyset \mid\!\sim_{\mathfrak{G}_\infty} \Gamma \rhd \varphi$.*

Proof

(a) Using induction on the length of proofs in L_∞, this follows immediately from 4.8(e) and 4.8(d).

(b) If $\emptyset \vdash_\infty \gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$ we can apply part (a) to the formula $\gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$, and by 4.8(c) we obtain $\emptyset \mid\!\sim_{\mathfrak{G}_\infty} \gamma_1 \wedge \cdots \wedge \gamma_n \rhd \varphi$. Using rule $(\wedge)$ we obtain $\emptyset \mid\!\sim_{\mathfrak{G}_\infty} \gamma_1, \ldots, \gamma_n \rhd \varphi$.

(c) If $\Gamma = \emptyset$ then the result follows from Lemma 3.2(b) and part (a). If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ then by Lemma 3.2(c) $\gamma_1, \ldots, \gamma_n \vdash^{\leqslant}_{\infty} \varphi$ if and only if $\emptyset \vdash_\infty \gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$, and the result follows from part (b). ∎

The previous results allow us to prove the equality between the two Gentzen systems we have been considering by a mixture of semantical and syntactical methods:

Theorem 4.10 $\mathfrak{G}_{L^{\leqslant}_\infty} = \mathfrak{G}_\infty$.

Proof By Lemma 4.6, the set of models of $\mathfrak{G}_{L^{\leqslant}_\infty}$ (which, by Theorem 4.3, is the set of all full g-models of $L^{\leqslant}_\infty$) is a subset of the set of models of $\mathfrak{G}_\infty$. Since any Gentzen system is complete with respect to its own models, we have that any derivation in $\mathfrak{G}_\infty$ also holds in $\mathfrak{G}_{L^{\leqslant}_\infty}$. The converse will be proved syntactically. We have to prove that all the initial sequents of $\mathfrak{G}_{L^{\leqslant}_\infty}$ are derivable sequents of $\mathfrak{G}_\infty$ and that all the rules of $\mathfrak{G}_{L^{\leqslant}_\infty}$ are derived rules of $\mathfrak{G}_\infty$. By Definition 4.2, the initial sequents of $\mathfrak{G}_{L^{\leqslant}_\infty}$ are sequents of the form $\emptyset \rhd \varphi$, where $\emptyset \vdash^{\leqslant}_{\infty} \varphi$, or of the form $\gamma_1, \ldots, \gamma_n \rhd \varphi$ with $\gamma_1, \ldots, \gamma_n \vdash^{\leqslant}_{\infty} \varphi$. In both cases Lemma 4.9(c) yields the desired result. The structural rules are present in both systems. And finally in $\mathfrak{G}_{L^{\leqslant}_\infty}$ we have the congruence rules. Since there are only two primitive connectives in the language, namely $\neg$ and $\to$, we only have to prove that:

(a) $\{x_1 \rhd y_1 , y_1 \rhd x_1\} \mid\!\sim_{\mathfrak{G}_\infty} \neg x_1 \rhd \neg y_1$.

(b) $\{x_1 \rhd y_1 , y_1 \rhd x_1 , x_2 \rhd y_2 , y_2 \rhd x_2\} \mid\!\sim_{\mathfrak{G}_\infty} x_1 \to x_2 \rhd y_1 \to y_2$.

The first one follows from rule $(\neg)$, and the second one is easily proved using (16) and (21) plus structural rules. ∎

The equality of those Gentzen systems and the properties of the selfextensional logics with conjunction make it possible to summarize the relationship between $\mathfrak{G}_\infty$ and $L^{\leqslant}_\infty$ in the following:

Corollary 4.11 *The Gentzen system $\mathfrak{G}_\infty$ satisfies the following properties:*

(a) *$\mathfrak{G}_\infty$ is algebraizable, and its largest equivalent algebraic semantics is the variety* $\mathbf{W}$.

(b) $\mathfrak{G}_\infty$ *is fully adequate for* $L_\infty^{\leqslant}$*, that is, the set of all models of* $\mathfrak{G}_\infty$ *is the set of all full g-models of* $L_\infty^{\leqslant}$*.*

(c) $L_\infty^{\leqslant}$ *is defined by the derivable sequents of* $\mathfrak{G}_\infty$*, that is the logic* $L_\infty^{\leqslant}$ *is characterized by the following property: for all* $\Gamma \cup \{\varphi\} \subseteq Fm$*,* $\Gamma \vdash_\infty^{\leqslant} \varphi$ *if and only if there exists a finite set* $\Gamma_0 \subseteq \Gamma$ *such that the sequent* $\Gamma_0 \rhd \varphi$ *is derivable in* $\mathfrak{G}_\infty$*.* ∎

One consequence of the above results is that we can characterize the deductive system $L_\infty^{\leqslant}$ by a set of properties that are very close to what Wójcicki calls "Tarski-style conditions" in [34, Section 2.3], that is, properties that express a relationship between the consequence and *one* logical connective (actually only (f) of Proposition 4.12 does not fall under Wójcicki's definition). This is achieved by formulating the property of being a model of a Gentzen system in terms of the closure operator:

Proposition 4.12 *Let* $\langle \boldsymbol{A}, \mathcal{C} \rangle$ *be an arbitrary g-matrix, with associated closure operator* C*. Then* $\langle \boldsymbol{A}, \mathcal{C} \rangle$ *is a model of* $\mathfrak{G}_\infty$ *if and only if it satisfies the following properties, for all* $a, b, c \in A$ *and all* $X \subseteq A$*:*

(a) Identity: $a \to a \in C(\emptyset)$.

(b) Conjunction: $C(a \wedge b) = C(a, b)$.

(c) Disjunction: $C(X, a \vee b) = C(X, a) \cap C(X, b)$.

(d) Double Negation: $C(\neg\neg a) = C(a)$.

(e) Contraposition: $C(a) \subseteq C(b) \Longrightarrow C(\neg b) \subseteq C(\neg a)$.

(f) Residuation: $c \in C(a * b) \Longleftrightarrow b \to c \in C(a)$.

(g) Premise permutation: $C\big(a \to (b \to c)\big) = C\big(b \to (a \to c)\big)$.

Proof Any g-matrix is a model of the structural rules and of the initial sequent called "axiom", and one can check that to be a model of the other initial sequents and rules in Definition 4.4 amounts exactly to satisfying the properties in the above list: The rules for $\to, \neg, *$ have been so chosen in order to make this trivial, while for $\wedge$ and $\vee$ this is easy to show, using rule $(\wedge)$ instead of $(\wedge\rhd)$; see for instance Theorem 2.13 of [15]. ∎

Any deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ can be presented as a generalized matrix $\langle \boldsymbol{Fm}_{\mathcal{L}}, \mathcal{T}h\mathcal{S} \rangle$ where $\mathcal{T}h\mathcal{S}$ is the collection of its ***theories*** (the sets of formulas closed under $\vdash_{\mathcal{S}}$). The associated closure operator $Cn_{\mathcal{S}}$ is given by the expression $\varphi \in Cn_{\mathcal{S}}(\Gamma) \Longleftrightarrow \Gamma \vdash_{\mathcal{S}} \varphi$. In this way it makes sense to say that a deductive system satisfies or does not satisfy properties like those above. When comparing several deductive systems over the same language, we say that $\langle \mathcal{L}, \vdash_1 \rangle$ is ***weaker*** than $\langle \mathcal{L}, \vdash_2 \rangle$ when the latter is an extension of the former in the sense explained in section 3, that is, when $\vdash_1 \subseteq \vdash_2$ as binary relations between sets of formulas and formulas, so that $\Gamma \vdash_1 \varphi$ implies $\Gamma \vdash_2 \varphi$. We then have:

Corollary 4.13 $L_\infty^{\leqslant}$ *is the weakest deductive system satisfying the properties in Proposition 4.12.*

Proof Each deductive system, when considered as a g-matrix, is obviously a g-model of itself, and indeed it is the weakest among all its g-models over the formula algebra, hence in particular it is also the weakest among all its full g-models over the formula algebra. But by Corollary 4.11(b) these full g-models are the models of $\mathfrak{G}_\infty$. Hence, the previous characterization of these models yields the desired result. ∎

To finish this section we will characterize *internal* and *external* deductive systems associated with the Gentzen system $\mathfrak{G}_\infty$ in the sense of [1]:

THEOREM 4.14 *The internal and external deductive systems associated with $\mathfrak{G}_\infty$ are $L_\infty^\leqslant$ and L_∞, respectively.*

PROOF By definition, $\langle \mathcal{L}, \vdash_i \rangle$ is the internal deductive system associated with $\mathfrak{G}_\infty$ when $\Gamma \vdash_i \varphi$ iff $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}_\infty} \Gamma \rhd \varphi$ for all finite Γ. Thus, by Corollary 4.11(a) $\langle \mathcal{L}, \vdash_i \rangle = L_\infty^\leqslant$. On the other hand, both the deductive system L_∞ and the Gentzen system $\mathfrak{G}_\infty$ are algebraizable with respect to the same class of algebras W (see Theorem 2.1(b) and Corollary 4.11). It happens that the equation $\varphi \approx 1$ is at the same time the translation of a formula φ in the first case, and the translation of the sequent $\emptyset \rhd \varphi$ in the second case. Therefore we have that

(27) $\{\varphi_1, \ldots, \varphi_n\} \vdash_\infty \varphi$ iff

(28) $\{\varphi_1 \approx 1, \ldots, \varphi_n \approx 1\} \models_{\mathsf{W}} \varphi \approx 1$ iff

(29) $\{\emptyset \rhd \varphi_1, \ldots, \emptyset \rhd \varphi_n\} \mathrel{|\!\!\sim}_{\mathfrak{G}_\infty} \emptyset \rhd \varphi.$

Now, the equivalence between (27) and (29) means that L_∞ is the external deductive system associated with $\mathfrak{G}_\infty$, according to AVRON's definition. ∎

5 The models of the Gentzen system

The fact that $\mathfrak{G}_\infty$ is fully adequate for $L_\infty^\leqslant$ has allowed us to obtain a first characterization of the full g-models of $L_\infty^\leqslant$. In this section we find other characterizations of the full g-models of both logics, $L_\infty^\leqslant$ and L_∞, from the study of the models of the Gentzen system $\mathfrak{G}_\infty$.

LEMMA 5.1 *Let $\varphi \approx \psi$ be one of the equations* (W1)–(W4) *that define the variety of Wajsberg algebras. Then the two sequents $\varphi \rhd \psi$ and $\psi \rhd \varphi$ are derivable in the Gentzen system $\mathfrak{G}_\infty$.*

PROOF See Appendix 6. ∎

With each g-matrix $\langle \boldsymbol{A}, C \rangle$ with closure operator C we can always associate a binary relation $\boldsymbol{\Lambda}(\mathcal{C})$ that is the abstract version of the interderivability relation of a deductive system: $\langle a, b \rangle \in \boldsymbol{\Lambda}(\mathcal{C}) \iff$ for every $T \in \mathcal{C}$, $a \in T \Leftrightarrow b \in T$; in terms of the associated closure operator C this amounts to $C(a) = C(b)$. In abstract algebraic logic this is called the ***Frege relation*** of the g-matrix. It is always an equivalence relation, but in general it need not be a congruence of the underlying algebra. There is always the largest congruence of $\boldsymbol{A}$ that is contained in $\boldsymbol{\Lambda}(\mathcal{C})$; it is denoted by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$ and is called the ***Tarski congruence*** of the g-matrix. The standard process to obtain reduced g-matrices is to factor out an arbitrary g-matrix by its Tarski congruence; the resulting g-matrix is usually denoted by $\langle \boldsymbol{A}^*, \mathcal{C}^* \rangle$, and the projection from $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ by π^*. This mapping is a strict surjective homomorphism, which means that it establishes an order isomorphism between the families $\mathcal{C}$ and $\mathcal{C}^*$ and a correspondence between the associated closure operators so that $\pi^* \circ C = C^* \circ \pi^*$.

In the best behaved situations the Frege relation is enough to perform this process:

LEMMA 5.2 *Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be an arbitrary g-matrix. Then the following conditions are equivalent:*

(i) *$\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a model of $\mathfrak{G}_\infty$, that is, a full g-model of $L_\infty^{\leqslant}$.*

(ii) *$\boldsymbol{\Lambda}(\mathcal{C})$ is a congruence on $\boldsymbol{A}$, $\boldsymbol{A}^* = \boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}) \in \mathbf{W}$ and $\mathcal{C}^* = \mathcal{C}/\boldsymbol{\Lambda}(\mathcal{C}) = \mathcal{F}_{\leqslant}(\boldsymbol{A}^*)$.*

PROOF Assume $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a model of $\mathfrak{G}_\infty$. Since $\mathfrak{G}_\infty = \mathfrak{G}_{L_\infty^{\leqslant}}$, $\langle \boldsymbol{A}, \mathcal{C} \rangle$ will be a model of the rules (cong) of Definition 4.2, and this easily implies that $\boldsymbol{\Lambda}(\mathcal{C})$ is a congruence, so $\boldsymbol{\Lambda}(\mathcal{C}) = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C})$ and then the g-matrix $\big\langle \boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}), \mathcal{C}/\boldsymbol{\Lambda}(\mathcal{C}) \big\rangle$ is reduced. But since $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full g-model of $L_\infty^{\leqslant}$, the g-matrix $\big\langle \boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}), \mathcal{C}/\boldsymbol{\Lambda}(\mathcal{C}) \big\rangle$ is a reduced full g-model of $L_\infty^{\leqslant}$, and by Corollary 3.7 it has to be of the form $\big\langle \boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}), \mathcal{F}_{\leqslant}\big(\boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C})\big) \big\rangle$ with $\boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}) \in \mathbf{W}$. This shows (ii).

To prove the converse note that again by Corollary 3.7, if $\boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}) \in \mathbf{W}$ then the g-matrix $\big\langle \boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C}), \mathcal{F}_{\leqslant}\big(\boldsymbol{A}/\boldsymbol{\Lambda}(\mathcal{C})\big) \big\rangle$ is a reduced full g-model of $\mathfrak{G}_\infty$, and hence by Corollary 4.11(b) it is a model of $\mathfrak{G}_\infty$. Since both properties are preserved under π^* and $\big(\pi^*\big)^{-1}$, this shows (i). ∎

We are going to see that the relations between the full g-models of $L_\infty^{\leqslant}$ and those of L_∞ mimmick several of the existing ones between the two deductive systems. As wwe have seen in Corollary 3.9, the theories of L_∞ are those of $L_\infty^{\leqslant}$ that are closed under Modus Ponens, or under other alternative rules. By comparing Theorem 2.1(e), Lemma 3.8 and Corollary 3.7 we see that this relation also holds between reduced full g-models of the two logics. Let us see that this extends to the non-reduced ones. To this end, for an arbitrary g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ we denote by $\mathcal{C}_{MP}$ the family of sets in $\mathcal{C}$ that are closed under Modus Ponens, that is $\mathcal{C}_{MP} = \{T \in \mathcal{C} : T \text{ is MP-closed}\}$. Then:

THEOREM 5.3 *A g-matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full g-model of L_∞ if and only if $\mathcal{C} = \mathcal{D}_{MP}$ for some closure system $\mathcal{D}$ such $\langle \boldsymbol{A}, \mathcal{D} \rangle$ is a full g-model of $L_\infty^{\leqslant}$.*

PROOF Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be a full g-model of L_∞. By Theorem 2.1(e), its reduction is $\langle \boldsymbol{A}^*, \mathcal{C}^* \rangle$ with $\boldsymbol{A}^* \in \mathbf{W}$ and $\mathcal{C}^* = \mathcal{F}_{\rightarrow}(\boldsymbol{A}^*)$. The g-matrix $\langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant}(\boldsymbol{A}^*) \rangle$ is a model of $\mathfrak{G}_\infty$. Since $\mathcal{F}_{\rightarrow}(\boldsymbol{A}^*) \subseteq \mathcal{F}_{\leqslant}(\boldsymbol{A}^*)$, $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}^*}\big(\mathcal{F}_{\leqslant}(\boldsymbol{A}^*)\big) \subseteq \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}^*}\big(\mathcal{F}_{\rightarrow}(\boldsymbol{A}^*)\big) = Id$, and therefore the g-matrix $\big\langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant}(\boldsymbol{A}^*) \big\rangle$ is reduced. Thus it is a reduced model of $\mathfrak{G}_\infty$ and hence a reduced full g-model of $L_\infty^{\leqslant}$. Now let us put $\mathcal{D} = (\pi^*)^{-1}\big[\mathcal{F}_{\leqslant}(\boldsymbol{A}^*)\big]$. Then the g-matrix $\langle \boldsymbol{A}, \mathcal{D} \rangle$ is a full g-model of $L_\infty^{\leqslant}$, and it is easy to show that $\mathcal{D}_{MP} = \mathcal{C}$, using that $\big(\mathcal{F}_{\leqslant}(\boldsymbol{A}^*)\big)_{MP} = \mathcal{F}_{\rightarrow}(\boldsymbol{A}^*)$.

For the converse, let us start from a g-matrix $\langle \boldsymbol{A}, \mathcal{D} \rangle$ that is a full g-model of $L_\infty^{\leqslant}$, and consider its reduction, which by Corollaries 3.7 and 4.11 and Lemma 5.2 has to be of the form $\langle \boldsymbol{A}^*, \mathcal{D}^* \rangle = \big\langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant}(\boldsymbol{A}^*) \big\rangle$ with $\boldsymbol{A}^* \in \mathbf{W}$. Thus by definition $(\mathcal{D}^*)_{MP}$ is exactly the set of lattice filters of $\boldsymbol{A}^*$ that are closed under Modus Ponens, that is, the implicative filters of $\boldsymbol{A}^*$. Therefore by Theorem 2.1(e)

$\langle \boldsymbol{A}^*, (\mathcal{D}^*)_{MP} \rangle$ is a full g-model of L_∞, and since $(\pi^*)^{-1}\big[(\mathcal{D}^*)_{MP}\big] = \mathcal{D}_{MP}$, the g-matrix $\langle \boldsymbol{A}, \mathcal{D}_{MP} \rangle$ is also a full g-model of L_∞. ∎

Now we show that the *graded deduction theorem* of L_∞ is also inherited by its full g-models:

PROPOSITION 5.4 *Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be a full g-model of $L_\infty^{\leqslant}$. Then for all $a_1, \ldots, a_n, a$ in A, $a \in C(a_1, \ldots, a_n)$ if and only if $a_1 \wedge \cdots \wedge a_n \to a \in C(\emptyset)$.*

PROOF As was recalled before, the reduction mapping π^* is a strict surjective homomorphism (bilogical morphism) from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto $\langle \boldsymbol{A}^*, \mathcal{C}^* \rangle = \langle \boldsymbol{A}^*, \mathcal{F}_{\leqslant} \rangle$. Then $a \in C(a_1, \ldots, a_n)$ iff $\pi^*(a) \in C^*\big(\pi^*(a_1), \ldots, \pi^*(a_n)\big) = F_{\leqslant}\big(\pi^*(a_1), \ldots, \pi^*(a_n)\big) = F_{\leqslant}\big(\pi^*(a_1) \wedge \cdots \wedge \pi^*(a_n)\big)$ iff $\pi^*(a_1) \wedge \cdots \wedge \pi^*(a_n) \leqslant \pi^*(a)$ iff $\pi^*(a_1 \wedge \cdots \wedge a_n \to a) = 1$ iff $a_1 \wedge \cdots \wedge a_n \to a \in C(\emptyset)$. ∎

From this, and due to the fact that all g-matrices we consider are finitary, we readily infer that the full g-models of $L_\infty^{\leqslant}$ are completely determined by their theorems:

COROLLARY 5.5 *If $\langle \boldsymbol{A}, \mathcal{C}_1 \rangle$ and $\langle \boldsymbol{A}, \mathcal{C}_2 \rangle$ are two full g-models of $L_\infty^{\leqslant}$ on the same algebra, and $C_1(\emptyset) = C_2(\emptyset)$, then $\mathcal{C}_1 = \mathcal{C}_2$.* ∎

THEOREM 5.6 *For an arbitrary algebra $\boldsymbol{A}$, the mapping $\langle \boldsymbol{A}, \mathcal{C} \rangle \longmapsto \langle \boldsymbol{A}, \mathcal{C}_{MP} \rangle$ establishes a bijection between the set of all full g-models of $L_\infty^{\leqslant}$ over $\boldsymbol{A}$ and the set of all full g-models of L_∞ over $\boldsymbol{A}$.*

PROOF By Theorem 5.3, the mapping is surjective. Now observe that if $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full g-model of $L_\infty^{\leqslant}$ then, by Lemma 4.8(d), its set of theorems $C(\emptyset)$ is closed under Modus Ponens. As a consequence, $C(\emptyset) \in \mathcal{C}_{MP}$ and hence $C(\emptyset) = C_{MP}(\emptyset)$. This observation and Corollary 5.5 imply that the mapping is one-to-one. Thus, it is a bijection. ∎

Generalizing what we did for deductive systems at the end of Section 4, we can give these sets of g-matrices an ordering structure by defining $\langle \boldsymbol{A}, \mathcal{C}_1 \rangle \leqslant \langle \boldsymbol{A}, \mathcal{C}_2 \rangle$ if and only if $C_1(X) \subseteq C_2(X)$ for every $X \subseteq A$. Then they become complete lattices and each of them is isomorphic to the lattice of all congruences θ of $\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \in \mathbf{W}$; see [11, Section 2.3] for details. It is also easy to see directly that the bijection of Theorem 5.6 is actually a lattice isomorphism between them.

We finish the section by expressing the relation between the full g-models of the two logics through the corresponding closure operators. We first show that one can find the intended relationship in Wajsberg algebras:

PROPOSITION 5.7 *Let $\boldsymbol{A} \in \mathbf{W}$ and $a \in A$. Then $F_{\rightarrow}(a) = \bigcup_{n \geqslant 1} F_{\leqslant}(a^n)$.*

PROOF Using the characterization (11) for $X = \emptyset$ we find that $b \in F_{\rightarrow}(a)$ iff there is some $n \geqslant 1$ such that $a^n \to b \in F_{\rightarrow}(\emptyset)$. But since $F_{\rightarrow}(\emptyset) = \{1\}$, to say that $a^n \to b \in F_{\rightarrow}(\emptyset)$ amounts to saying that $a^n \leqslant b$ or, using (10), that $b \in F_{\leqslant}(a^n)$. ∎

Theorem 5.8 *Let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be a full g-model of $L_\infty^\leqslant$, with associated closure operator C. Then $\langle \boldsymbol{A}, \mathcal{C}_{MP} \rangle$, the corresponding full g-model of L_∞, is characterized by the following properties of its associated closure operator C_{MP}:*

(1) $C_{MP}(\emptyset) = C(\emptyset)$.

(2) *For any* $a_1, \dots, a_k \in A$, $C_{MP}(a_1, \dots, a_k) = \bigcup_{n \geqslant 1} C\big((a_1 \wedge \cdots \wedge a_k)^n\big)$.

(3) C_{MP} *is finitary.*

Proof As observed in the proof of Theorem 5.6, part (1) is a consequence of Lemma 4.8(d), plus the fact that the full g-models of $L_\infty^\leqslant$ are the models of $\mathfrak{G}_\infty$. Part (3) is obvious. Now, $\langle \boldsymbol{A}, \mathcal{C} \rangle$ satisfies conjunction, and since $\mathcal{C}_{MP} \subseteq \mathcal{C}$, also $\langle \boldsymbol{A}, \mathcal{C}_{MP} \rangle$ satisfies it, therefore $C_{MP}(a_1, \dots, a_k) = C_{MP}(a_1 \wedge \cdots \wedge a_k)$. As a consequence, in order to prove part (2) it is enough to prove that for a single $a \in A$, $C_{MP}(a) = \bigcup_{n \geqslant 1} C(a^n)$. For this we use the reduction process which establishes a bilogical morphism π^* between $\langle \boldsymbol{A}, \mathcal{C} \rangle$ and $\big\langle \boldsymbol{A}^*, \mathcal{F}_\leqslant(\boldsymbol{A}^*) \big\rangle$. This bilogical morphism is a lattice isomorphism between the closed-set systems $\mathcal{C}$ and $\mathcal{F}_\leqslant(\boldsymbol{A}^*)$. Since $\mathcal{C}_{MP} \subseteq \mathcal{C}$, it is also an isomorphism between $\mathcal{C}_{MP}$ and $\mathcal{F}_\rightarrow(\boldsymbol{A}^*)$, thus π^* is a bilogical morphism between $\langle \boldsymbol{A}, \mathcal{C}_{MP} \rangle$ and $\big\langle \boldsymbol{A}^*, \mathcal{F}_\rightarrow(\boldsymbol{A}^*) \big\rangle$. Since $\boldsymbol{A} \in \mathsf{W}$, we can use Proposition 5.7, and the following equivalences are then obvious: $b \in C_{MP}(a)$ iff $\pi^*(b) \in F_\rightarrow\big(\pi^*(a)\big)$ iff there is some $n \geqslant 1$ such that $\pi^*(b) \in F_\leqslant\big(\pi^*(a^n)\big)$ iff there is some $n \geqslant 1$ such that $b \in C(a^n)$. ∎

6 Conclusions and open problems

We have defined an infinite-valued logic $L_\infty^\leqslant$ by preservation of degrees of truth, where the truth-values come from the real unit interval, and we have described its relationships with the usual infinite-valued Łukasiewicz logic L_∞ and with the variety of Wajsberg algebras and their lattice structure. Some selected results are:

- While L_∞ is algebraizable, $L_\infty^\leqslant$ is not. In fact, it is not even protoalgebraic, which implies that most of the constructions and results of the traditional algebraic logic theory cannot be applied to it.
- $L_\infty^\leqslant$ is selfextensional, again in contrast with L_∞ which is not.
- The two logics have the same algebraic counterpart, the variety of Wajsberg algebras, but on these algebras the matrix models of L_∞ are given by the implicative filters while those of $L_\infty^\leqslant$ are given by the lattice filters.

We have presented a finite Gentzen-style calculus that can be interpreted as a calculus of its valid entailments, and this has given a characterization of this logic by a set of abstract properties of its consequence operator. We have described its full generalized models in several ways, and the full generalized models of L_∞ as well. We have shown how several of the abstract properties that relate $L_\infty^\leqslant$ to L_∞ are also inherited in the relation between their respective full generalized models.

The main open problem is to find a Hilbert-style axiomatization of $L_\infty^\leqslant$. We remark that Modus Pones is not a rule of this logic. Indeed, we have shown that L_∞ is exactly the purely inferential extension of $L_\infty^{\leqslant}$ obtained by adding the rule of Modus Ponens to it. Since all known presentations of L_∞ take this rule as their only inference rule, we presume that this open problem must be a difficult one.

Another open problem raised by this paper is of a general kind, and belongs to abstract algebraic logic. In Proposition 5.4 we show that the full generalized models of $L_\infty^{\leqslant}$ inherit the metalogical property of the logic called the graded deduction theorem. This raises the question of investigating whether this property is always inherited by the full generalized models of a logic that has it (as it happens with the ordinary deduction theorem), or to find under what conditions this is true.

Appendix: Proofs of some lemmas

Proof of Lemma 4.7

(16) Apply (res2) to $\xi \to \varphi \rhd \xi \to \varphi$, then (cut) with $\varphi \rhd \psi$ and finally (res1).

(17) Apply (res1) to $\psi * \xi \rhd \xi \to \psi$, then (cut) with $\varphi \rhd \psi$ and finally (res2).

(18)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \rhd \psi}{\psi \rhd \psi \lor \neg\varphi}\;(\rhd\lor)}{\neg(\psi \lor \neg\varphi) \rhd \neg\psi}\;(\neg)}{\neg\big((\psi \to \neg\varphi) \to \neg\varphi\big) \rhd \neg\psi}\;(\mathrm{def}\lor)}{(\psi \to \neg\varphi) * \varphi \rhd \neg\psi}\;(\mathrm{def}\,*)}{\psi \to \neg\varphi \rhd \varphi \to \neg\psi}\;(\mathrm{res1})$$

(19)

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\psi \to \neg\varphi \rhd \varphi \to \neg\psi}\;(18)}{(\psi \to \neg\varphi) * \neg\xi \rhd (\varphi \to \neg\psi) * \neg\xi}\;(17)}{\neg((\varphi \to \neg\psi) * \neg\xi) \rhd \neg\big((\psi \to \neg\varphi) * \neg\xi\big)}\;(\neg)}{(\varphi \to \neg\psi) \to \xi \rhd (\psi \to \neg\varphi) \to \xi}\;(\mathrm{def}\,*)$$

(20) Start with

$$\dfrac{\dfrac{\dfrac{\varphi \rhd \varphi}{\varphi \rhd \varphi \lor \neg\psi}\;(\rhd\lor)}{\varphi \rhd (\varphi \to \neg\psi) \to \neg\psi}\;(\mathrm{def}\lor) \qquad \dfrac{\emptyset}{(\varphi \to \neg\psi) \to \neg\psi \rhd (\psi \to \neg\varphi) \to \neg\psi}\;(19)}{\varphi \rhd (\psi \to \neg\varphi) \to \neg\psi}\;(\mathrm{cut})$$

[

and continue the proof with

$$
\dfrac{\dfrac{\dfrac{\varphi \rhd (\psi \to \neg\varphi) \to \neg\psi \qquad \dfrac{\emptyset}{(\psi \to \neg\varphi) \to \neg\psi \rhd \psi \to \neg(\psi \to \neg\varphi)}\,(18)}{\varphi \rhd \psi \to \neg(\psi \to \neg\varphi)}\,(\text{cut})}{\varphi \rhd \psi \to \psi * \varphi}\,(\text{def}*)}{\varphi * \psi \rhd \psi * \varphi}\,(\text{res2})
$$

(21)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \rhd \psi \qquad \dfrac{\psi \rhd \psi}{\psi \rhd \psi \vee \xi}\,(\rhd\vee)}{\varphi \rhd \psi \vee \xi}\,(\text{cut})}{\varphi \rhd (\psi \to \xi) \to \xi}\,(\text{def}\vee)}{\varphi * (\psi \to \xi) \rhd \xi}\,(\text{res2}) \qquad \dfrac{\emptyset}{(\psi \to \xi) * \varphi \rhd \varphi * (\psi \to \xi)}\,(20)}{(\psi \to \xi) * \varphi \rhd \xi}\,(\text{cut})}{\psi \to \xi \rhd \varphi \to \xi}\,(\text{res2})
$$

(22)

$$
\dfrac{\dfrac{\dfrac{\psi \rhd \psi}{\psi \rhd \neg\neg\psi}\,(\rhd\neg\neg)}{\varphi \to \psi \rhd \varphi \to \neg\neg\psi}\,(16) \qquad \dfrac{\dfrac{\psi \rhd \psi}{\psi \rhd \neg\neg\psi}\,(\rhd\neg\neg)}{\varphi \to \neg\neg\psi \rhd \neg\psi \rhd \neg\varphi}\,(16)}{\varphi \to \psi \rhd \neg\psi \to \neg\varphi}\,(\text{cut})
$$

$$
\dfrac{\dfrac{\dfrac{\psi \rhd \psi}{\neg\neg\psi \rhd \psi}\,(\neg\neg\rhd)}{\varphi \to \neg\neg\psi \rhd \varphi \to \psi}\,(16) \qquad \dfrac{\emptyset}{\neg\psi \to \neg\varphi \rhd \varphi \to \neg\neg\psi}\,(18)}{\neg\psi \to \neg\varphi \rhd \varphi \to \psi}\,(\text{cut})
$$

(23)

$$
\dfrac{\dfrac{\dfrac{\varphi \rhd \varphi}{\varphi, \psi \rhd \varphi}\,(\text{w}) \qquad \dfrac{\psi \rhd \psi}{\varphi, \psi \rhd \psi}\,(\text{w})}{\varphi, \psi \rhd \psi \wedge \varphi}\,(\rhd\wedge)}{\varphi \wedge \psi \rhd \psi \wedge \varphi}\,(\wedge)
$$

(24)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\varphi \to \psi \rhd \neg\psi \to \neg\varphi}\,(22)}{(\varphi \to \psi) \to \neg\varphi \rhd (\neg\psi \to \neg\varphi) \to \neg\varphi}\,(21)}{\neg\neg\big((\varphi \to \psi) \to \neg\varphi\big) \rhd (\neg\psi \to \neg\varphi) \to \neg\varphi}\,(\neg\neg\rhd)}{\neg((\varphi \to \psi) * \varphi) \rhd \neg\psi \vee \neg\varphi}\,(\text{def}\,*, \vee)}{\neg(\neg\psi \vee \neg\varphi) \rhd \neg\neg((\varphi \to \psi) * \varphi)}\,(\neg) \qquad \dfrac{(\varphi \to \psi) * \varphi \rhd (\varphi \to \psi) * \varphi}{\neg\neg\big((\varphi \to \psi) * \varphi\big) \rhd (\varphi \to \psi) * \varphi}\,(\neg\neg\rhd)}{\dfrac{\neg(\neg\psi \vee \neg\varphi) \rhd (\varphi \to \psi) * \varphi}{\dfrac{\psi \wedge \varphi \rhd (\varphi \to \psi) * \varphi}{\varphi \wedge \psi \rhd (\varphi \to \psi) * \varphi}\,(23) + (\text{cut})}\,(\text{def}\wedge)}\,(\text{cut})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\psi \to \neg\varphi \rhd \varphi \to \psi}{(\neg\psi \to \neg\varphi) \to \neg\varphi \rhd (\varphi \to \psi) \to \neg\varphi}\,(22) + (21)}{\neg\psi \vee \neg\varphi \rhd (\varphi \to \psi) \to \neg\varphi}\,(\text{def}\vee)}{\neg\big((\varphi \to \psi) \to \neg\varphi\big) \rhd \neg(\neg\psi \vee \neg\varphi)}\,(\neg)}{(\varphi \to \psi) * \varphi \rhd \psi \wedge \varphi}\,(\text{def}\,*, \wedge) \qquad \dfrac{\emptyset}{\psi \wedge \varphi \rhd \varphi \wedge \psi}\,(23)}{(\varphi \to \psi) * \varphi \rhd \varphi \wedge \psi}\,(\text{cut})$$

(25)

$$\dfrac{\dfrac{\emptyset}{\varphi \to \varphi \rhd \psi \to \psi}\,(\text{id}{\to}) + (\text{w})}{(\varphi \to \varphi) * \psi \rhd \psi}\,(\text{res2})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\psi \to \psi \rhd \varphi \to \varphi}\,(\text{id}{\to}) + (\text{w})}{(\psi \to \psi) * \psi \rhd (\varphi \to \varphi) * \psi}\,(17)}{(\psi \wedge \psi) \rhd (\varphi \to \varphi) * \psi}\,(24) + (\text{cut}) \qquad \dfrac{\psi \rhd \psi \quad \psi \rhd \psi}{\psi \rhd \psi \wedge \psi}\,(\rhd\wedge)}{\psi \rhd (\varphi \to \varphi) * \psi}\,(\text{cut})$$

(26) Apply $(\rhd\vee)$ to $\varphi \rhd \varphi$ and to $\psi \rhd \psi$ to obtain $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}_\infty} \varphi \rhd \varphi \vee \psi$ and $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}_\infty} \psi \rhd \varphi \vee \psi$. From these results, using $(\rhd\vee)$ and by symmetry we obtain that $\emptyset \mathrel{|\!\!\sim}_{\mathfrak{G}_\infty} \varphi \vee \psi \lhd\rhd \psi \vee \varphi$.

Proof of Lemma 4.8

(a)

$$\dfrac{\emptyset \rhd \varphi \to \varphi \qquad \dfrac{\dfrac{\dfrac{\emptyset \rhd \varphi \to \varphi}{\varphi \wedge \varphi \rhd \varphi \to \varphi}\,(\text{w}) \qquad \dfrac{\emptyset}{(\varphi \to \varphi) * \varphi \rhd \varphi \wedge \varphi}\,(24)}{(\varphi \to \varphi) * \varphi \rhd \varphi \to \varphi}\,(\text{cut})}{\varphi \to \varphi \rhd \varphi \to (\varphi \to \varphi)}\,(\text{res1})}{\emptyset \rhd \varphi \to (\varphi \to \varphi)}\,(\text{cut})$$

(b) Start the proof with

$$\dfrac{\dfrac{\dfrac{\emptyset \rhd \varphi \quad \emptyset \rhd \varphi}{\emptyset \rhd \varphi \wedge \varphi}\,(\rhd\wedge) \qquad \dfrac{\emptyset}{\varphi \wedge \varphi \rhd (\varphi \to \varphi) * \varphi}\,(24)}{\emptyset \rhd (\varphi \to \varphi) * \varphi}\,(\text{cut}) \qquad \dfrac{\dfrac{\emptyset \rhd \psi}{\varphi \rhd \psi}\,(\text{w})}{(\varphi \to \varphi) * \varphi \rhd (\varphi \to \varphi) * \psi}\,(17)+(20)}{\emptyset \rhd (\varphi \to \varphi) * \psi}\,(\text{cut})$$

and then finish it with

$$\dfrac{\emptyset \rhd (\varphi \to \varphi) * \psi \qquad \dfrac{\dfrac{\emptyset \rhd \varphi}{\varphi \to \varphi \rhd \varphi}\,(\text{w})}{(\varphi \to \varphi) * \psi \rhd \varphi * \psi}\,(17)}{\emptyset \rhd \varphi * \psi}\,(\text{cut})$$

(c) $(\Rightarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset \rhd \varphi \to \psi}{\varphi \to (\varphi \to \varphi) \rhd \varphi \to \psi}\,(\text{w})}{\big(\varphi \to (\varphi \to \varphi)\big) * \varphi \rhd \psi}\,(\text{res2})}{\varphi \wedge (\varphi \to \varphi) \rhd \psi}\,(24)+(\text{cut}) \qquad \dfrac{\varphi \rhd \varphi}{\varphi \wedge (\varphi \to \varphi) \rhd \varphi}\,(\wedge\rhd)}{\varphi \rhd \psi}\,(\text{cut})$$

$(\Leftarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \rhd \psi \qquad \dfrac{\varphi \rhd \varphi}{\varphi \wedge (\varphi \to \varphi) \rhd \varphi}\,(\wedge\rhd)}{\varphi \wedge (\varphi \to \varphi) \rhd \psi}\,(\text{cut})}{\big(\varphi \to (\varphi \to \varphi)\big) * \varphi \rhd \psi}\,(24)+(\text{cut}) \qquad \dfrac{\emptyset}{\emptyset \rhd \varphi \to (\varphi \to \varphi)}\,(\text{a})}{\varphi \to (\varphi \to \varphi) \rhd \varphi \to \psi}\,(\text{res1})}{\emptyset \rhd \varphi \to \psi}\,(\text{cut})$$

(d)

$$\dfrac{\dfrac{\dfrac{\emptyset \rhd \varphi \quad \emptyset \rhd \varphi \to \psi}{\emptyset \rhd \varphi * (\varphi \to \psi)}\,(\text{b})}{\emptyset \rhd \varphi \wedge \psi}\,(24)+(\text{cut}) \qquad \dfrac{\varphi \rhd \varphi}{\varphi \wedge \psi \rhd \psi}\,(\wedge\rhd)}{\emptyset \rhd \psi}\,(\text{cut})$$

(e) We have to show the four axioms of $Ł_\infty$ (see Section 2).

(A1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\psi \rhd \varphi \to \varphi}\,(\text{id}\!\to)+(\text{w})}{\psi * \varphi \rhd \varphi}\,(\text{res2})}{\varphi * \psi \rhd \varphi}\,(20)+(\text{cut})}{\varphi \rhd \psi \to \varphi}\,(\text{res1})}{\emptyset \rhd \varphi \to (\psi \to \varphi)}\,(\text{c})$$

(A2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi \rhd \psi}{\psi \rhd \psi \vee \xi}\,(\rhd\vee)}{\varphi \to \psi \rhd \varphi \to (\psi \vee \xi)}\,(16)}{\varphi \to \psi \rhd \varphi \to \big((\psi \to \xi) \to \xi\big)}\,(\mathrm{def}\vee)}{\varphi \to \psi \rhd (\psi \to \xi) \to (\varphi \to \xi)}\,(\mathrm{ex}{\to})}{(\varphi \to \psi) \to \big((\psi \to \xi) \to (\varphi \to \xi)\big)}\,(\mathrm{c})$$

(A3) Straightforward by the definition of $\vee$ and (26).

(A4) Straightforward by (22) and (cut).

Proof of Lemma 5.1

We have to derive two sequents for each of the four identities that define the variety of Wajsberg algebras, see Section 2.

(W1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{\emptyset \rhd y \to y}\,(\mathrm{id}{\to})}{x \to x \rhd y \to y}\,(\mathrm{w})}{(x \to x) * y \rhd y}\,(\mathrm{res2})}{y * (x \to x) \rhd y}\,(20)}{y \rhd (x \to x) \to y}\,(\mathrm{res1})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{x \to x \rhd y \to y}\,(\mathrm{id}{\to}) + (\mathrm{w})}{(x \to x) \to y \rhd (y \to y) \to y}\,(21)}{(x \to x) \to y \rhd y \vee y}\,(\mathrm{def}\vee) \qquad \dfrac{y \rhd y \quad y \rhd y}{y \vee y \rhd y}\,(\vee\rhd)}{(x \to x) \to y \rhd y}\,(\mathrm{cut})$$

(W2) The sequent $(x \to y) \to \big((y \to z) \to (x \to z)\big) \rhd (x \to x)$ results from (id$\to$) and (w).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{y \rhd y \vee y}\,(26)}{x \to y \rhd x \to (y \vee z)}\,(16)}{(x \to y) \rhd x \to \big((y \to z) \to z\big)}\,(\mathrm{def}\vee)}{(x \to y) \rhd (y \to z) \to (x \to z)}\,(\mathrm{ex}{\to}) + (\mathrm{cut}) \qquad \dfrac{\dfrac{\dfrac{\emptyset}{\emptyset \rhd (x \to y) \to (x \to y)}\,(\mathrm{id}{\to})}{x \to x \rhd (x \to y) \to (x \to y)}\,(\mathrm{w})}{(x \to x) * (x \to y) \rhd (x \to y)}\,(\mathrm{res2})}{(x \to x) * (x \to y) \rhd \big((y \to z) \to (x \to z)\big)}\,(\mathrm{cut})}{(x \to x) \rhd (x \to y) \to \big((y \to z) \to (x \to z)\big)}\,(\mathrm{res1})$$

(W3)

$$\dfrac{\dfrac{\emptyset}{x \vee y \rhd y \vee x}\,(26)}{(x \to y) \to y \rhd \big((y \to x) \to x\big)}\,(\mathrm{def}\vee)$$

(W4) The sequent $(\neg x \to \neg y) \to (y \to x) \rhd (x \to x)$ results from (id$\to$) and (w).

$$\dfrac{\dfrac{\dfrac{\emptyset}{\neg x \to \neg y \rhd y \to x}\ (22)+(\mathrm{cut}) \qquad \dfrac{\emptyset}{(x \to x) * (\neg x \to \neg y) \rhd \neg x \to \neg y}\ (25)}{(x \to x) * (\neg x \to \neg y) \rhd (y \to x)}\ (\mathrm{cut})}{(x \to x) \rhd (\neg x \to \neg y) \to (y \to x)}\ (\mathrm{res1})$$

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Josep Maria Font, Antoni Torrens, Ventura Verdú
Department of Probability, Logic, and Statistics
Faculty of Mathematics, University of Barcelona
Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
{jmfont,v.verdu,atorrens}@ub.edu

Àngel J. Gil
Departament d'Economia i Empresa
Universitat Pompeu Fabra
C. Ramon Trias Fargas 27, 08005 Barcelona, Spain
angel.gil@upf.edu