

On substructural logics preserving degrees of truth

Josep Maria Font

*Dedicated to Professor Hirokira Ono,
master, colleague and friend.*

The purpose of this paper is to discuss how some ideas coming from the many-valued logic world can be introduced in a sensible way into the world of substructural logic; namely, the ideas around what does it mean for a logic to say that it *preserves degrees of truth*. The two mentioned subject areas are by their origin rather far apart; I would like to exemplify how the recent evolution of research in the field of substructural logics, and the application of central techniques from abstract algebraic logic, has revealed such borderline issues and has facilitated their investigation.

Degrees of truth are of course ubiquitous in the literature on many-valued and fuzzy logic, as one of the interpretations of non-classical truth-values. However, even logics admitting semantics with more than two degrees of truth are in general cast to preserve just one of them, “absolute” truth. Less discussed is the idea of a logic “preserving degrees of truth”. It was considered, in algebraic terms, in [14] and in [19]¹. Both assume an ordering relation between degrees of truth, so that the idea appears as related only to ordered algebras. Here I would like to give it a broader spectrum of application.

The idea of a logic preserving degrees of truth is often presented as opposed to that of a *truth-preserving logic*; I will try to demonstrate

¹Warning: in this book the term “truth-preserving” is used in the sense explained in (2) below, except on page 345 where it means “preserving degrees of truth” (and at that place “preserving validity” is used as a replacement for “truth-preserving”).

that such opposition may be viewed as only apparent. The idea of a truth-preserving logic is related to the classical conception of *logical consequence*, according to which

a conclusion follows logically from some premises if and only if, *whenever* the premises are *true*, the conclusion is also *true*. (1)

I think that the difference between the ideas of preserving truth and preserving degrees of truth does not lie in the acceptance or rejection of this conception of consequence, but in applying it to a different conception of the *true* in (1), which may consequently change the interpretation of the *whenever* in (1).

For an arbitrary algebraic (i.e., sentential) language, let \mathbf{Fm} denote the algebra of formulas of its similarity type. I identify a logic \mathcal{L} with its relation of consequence $\vdash_{\mathcal{L}}$. The (algebraic version of the) classical idea of a *truth structure* is a pair $\langle \mathbf{A}, T \rangle$ where \mathbf{A} is an algebra of the same similarity type with universe A , and $T \subseteq A$ is the *truth set*; traditionally this is called a *logical matrix*. A logic \mathcal{L} is ***the truth-preserving logic with respect to the structure*** $\langle \mathbf{A}, T \rangle$ when it satisfies, for all $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \quad (2)$$

$$\text{if } v(\gamma) \in T \ \forall \gamma \in \Gamma \text{ then } v(\varphi) \in T.$$

A logic \mathcal{L} ***preserves truth*** with respect to the same structure when it satisfies the part \Rightarrow in (2). Everyone knows that classical logic is the truth-preserving logic with respect to the structure $\langle \mathbf{2}, \{1\} \rangle$, and that ŁUKASIEWICZ's infinite-valued logic is the truth-preserving logic with respect to the structure $\langle [\mathbf{0}, \mathbf{1}], \{1\} \rangle$, where $[\mathbf{0}, \mathbf{1}]$ is ŁUKASIEWICZ's algebra on the real unit interval $[0,1]$. There is a finitariness problem though: A logic defined by (2) need not in general be finitary, and indeed if we define ŁUKASIEWICZ's logic in this way it is not; thus, it makes sense, depending on the context, to limit the usage of (2) as a definition to finite Γ , and force \mathcal{L} to be finitary by stipulating, for an infinite Γ , that

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \text{there is a finite } \Gamma_0 \subseteq \Gamma \text{ such that } \Gamma_0 \vdash_{\mathcal{L}} \varphi. \quad (3)$$

If one "believes" in only one truth structure, then the *whenever* in (1) is effected in (2) by the universal quantifier over all interpretations into this

structure, that is, all homomorphisms (also called *valuations*) from the algebra of formulas into the algebra \mathbf{A} supporting the truth structure. In this situation the *whenever* will just mean “for all interpretations in \mathbf{A} ”.

For the sake of pure generalization, and to accomodate other commitments about the nature of truth, one can extend the above expressions by considering a family of truth structures $\{\langle \mathbf{A}_i, T_i \rangle : i \in I\}$ and just adding a “ $\forall i \in I$ ” at the appropriate place; the *whenever* would then mean “for all interpretations in all truth structures in the family”. In algebraic logic tradition, (2) describes the logic defined by a matrix or (in the extended case) a class of matrices. As a particular case, when all \mathbf{A}_i are equal the family $\{\langle \mathbf{A}, T_i \rangle : i \in I\}$ has been called a *bundle of matrices*. Semantically equivalent objects are the so-called *ramified matrices* or *generalized matrices* [6], which are pairs of the form $\langle \mathbf{A}, \{T_i : I \in I\} \rangle$ where the family $\{T_i : I \in I\} \subseteq \mathcal{P}(A)$ is either arbitrary or required to be closed under intersections (semantically this makes no difference).

Most discussions about truth degrees tacitly or explicitly identify truth degrees with truth values, when there are more than two of these. My proposal is not to do so. I propose to view a generalized matrix $\langle \mathbf{A}, \{T_i : i \in I\} \rangle$ as a *structure of degrees of truth* in the following sense. Let us consider as *truth values* all values that formulas can take under an arbitrary evaluation, that is, all points in the algebra \mathbf{A} ; thus, for each $v \in \text{Hom}(\mathbf{Fm}, \mathbf{A})$ and each $\varphi \in \mathbf{Fm}$, $v(\varphi)$ is the truth value of the formula φ under the evaluation v . By contrast, each subset T_i represents a *degree of truth*. This opens the possibility that, for instance, the truth values are numerically precise values while the degrees correspond to certain ranges of values, or, in general, to groupings of values according to some criteria, so that the degrees can have a more qualitative contents than the values.

Now, in parallel with (2), a logic \mathcal{L} is ***the logic that preserves degrees of truth with respect to the structure*** $\langle \mathbf{A}, \{T_i : i \in I\} \rangle$ when it satisfies, for all (finite) $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$,

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \forall i \in I, \quad (4)$$

$$\text{if } v(\gamma) \in T_i \forall \gamma \in \Gamma \text{ then } v(\varphi) \in T_i.$$

Formally, this amounts to \mathcal{L} preserving truth with respect to all degrees in the sense of (2); but what has changed is the interpretation of the

role played by the subsets T_i : instead of thinking each one as giving a model of truth or truth structure, we understand them as representing a degree of truth, and *collectively* defining a structure of degrees of truth. A logic \mathcal{L} is said to **preserve degrees of truth** with respect to that structure when it satisfies part \Rightarrow of (4). These definitions can still be read as implementing the idea in (1), if the *true* in it is interpreted as “to have a certain degree of truth”, so that the *whenever* means “for all evaluations and all degrees of truth”: Thus, an entailment preserves degrees of truth if,

$$\begin{aligned} & \textit{whenever all the premises attain a certain degree} \\ & \textit{of truth, the conclusion also attains it.} \end{aligned} \quad (5)$$

Notice that by putting $\Gamma = \emptyset$ in (4) we find

$$\emptyset \vdash_{\mathcal{L}} \varphi \iff \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), v(\varphi) \in \bigcap_{i \in I} T_i. \quad (6)$$

Thus, the *theorems* or *tautologies* of the logic that preserves degrees of truth as in (4) coincide with those of the truth-preserving logic with respect to the structure $\langle \mathbf{A}, \bigcap_{i \in I} T_i \rangle$. The subset $\bigcap_{i \in I} T_i$ thus represents *absolute truth*. Incidentally, this is one of the reasons to demand the family $\{T_i : i \in I\}$ to be closed under intersections.

Now, how all this fits into the world of substructural logics? This investigation would have not been possible without the recent advancements in the algebraic study of substructural logics, and particularly with the applications of techniques of abstract algebraic logic. From the beginning of the topic and particularly in the recent years Professor Ono has been a leading figure, and his contributions have considerably influenced a turn in the direction of research on substructural logics.

The starting intuitions about substructural logics were mainly proof-theoretical. The first answer to the question

$$\textit{What is a substructural logic?} \quad (7)$$

was given by the creator of the term, KOSTA DOŠEN. He opened his historical introduction [4] to the foundational volume [5] by saying:

$$\begin{aligned} & \text{“The common denominator of several important non-} \\ & \text{classical logics is that in their sequent formulation they} \\ & \text{reject or restrict some of GENTZEN’s structural rules,} \\ & \text{[...]} \text{”} \end{aligned} \quad (8)$$

and in the second paragraph he declares:

“We shall call logics that can be obtained in this manner,
by restricting structural rules, *substructural logics*.” (9)

Let me emphasize two apparently innocent phrases in these fragments: “*in their sequent formulation*” and “*can be obtained*”. They seem to suggest that substructurality does not reside exactly in the logic but in the way it is obtained or formulated, so that substructurality is not a property inherent in a logic, but just in (some of) its formulations. An alternative view is to search the root of substructurality in some object inherently associated with the logic, so that it really becomes an *inherent* property of the logic. An object that can play this role is any class of algebras that is the algebraic counterpart of the logic according to some general theory of the algebraization of logics.

This alternative view is confirmed by the answer that question (7) has received after some years, an answer which does not conflict with (9) but rather perfects or complements it. The research on substructural logics has developed in several directions, in particular much research has been done on their semantics, cf. ONO’s paper [15] in the same volume [5]. A large group of researchers has focused on the study of residuated lattices and their relation to substructural logics. In an important survey paper [16] published ten years later than [5], ONO starts his conclusions by saying that

“Though there is already much literature on substructural logics, we have no common understanding of the definition of substructural logics.” (10)

and he finally proposes a definition:

“Substructural logics are the logics of residuated lattices.” (11)

This conception allows him to cover a much larger family of logics than those arising from (9), and to treat them in a unified way. In order to fully understand it, one should however have a definite meaning for the expression *the logic of* referred to a certain class of algebraic structures (as residuated lattices are). Confronting with the background work ONO refers to, most of which has been gathered in [11], we find implicit some elements that conform this meaning:

- (i) When using the expression *the logics of*, it is assumed that substructural logics are (sentential) *logics* in the ordinary sense, that is, substitution-invariant² relations of consequence on the algebra of formulas as defined by TARSKI, ŁOŚ and SUSZKO [18, 13]. This facilitates placing their study into more general frameworks, such as those of abstract algebraic logic.
- (ii) An apparent drawback of the previous fact is that, by definition, such consequence relations admit the structural rules. However, the conflict with (9) should be only apparent, as these consequences still do have *formulations* in terms of GENTZEN systems with restricted structural rules, which are also studied in [11]. Actually, the general theory of the relations between sentential logics and GENTZEN systems has made some significant advances inspired in the work done in substructural logics, see [17].
- (iii) The expression *the logic of* refers to the result of applying some general framework or scheme governing the relations between a logic and a class of algebras, such that for each class of algebras (of a certain kind) it automatically yields *its* logic, and conversely such that for each logic it yields *its* algebraic counterpart, a class of algebras inherently associated with the logic independently of its formulation.

While the general theory of abstract algebraic logic shows that no such universally applicable scheme exists, it also provides us with the notion of an algebraizable logic, which turns out to apply to the case of [16]. The theory of *algebraizability* of logics was formulated by BLOK and PIGOZZI in their seminal monograph [1], and was later extended and generalized by other people, see [3, 9]. It is this paradigm that requires the logical objects to be consequence relations rather than sets of formulas (theorems); however, if one considers only axiomatic extensions of a certain basic logic (in [11] it is the logic corresponding to the *full LAMBEK calculus*), then the exposition can be simplified by considering only the corresponding sets of theorems, since these characterize the extensions. Algebraically, this means we are only

²In traditional algebraic logic literature the term “structural” is used to mean the same as “substitution-invariant”, but in the context of substructural logics the latter term is clearly preferable.

dealing with subvarieties of a basic class of algebras, which can be taken as the variety of residuated lattices or the variety of FL-algebras.

- (iv) Among the features of the paradigm of algebraizability, I would like to emphasize here that it requires some logical connectives, or algebraic operations, notably fusion and implication, to play an already pre-designed role in the relations between the logic and the class of algebras; for instance, algebraizability needs *equivalence formulas* and *defining equations*, and these will forcefully use some of these connectives. This feature may be regarded by some as a drawback, as it somehow forces the interpretation of some of the logical symbols.

Let's be more concrete, following [11, pp. 92 ff]. **Residuated lattices** are algebras $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ of type $\langle 2, 2, 2, 2, 2, 0 \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, 1 \rangle$ is a monoid, and the operation \cdot (fusion) is both left- and right-residuated, with residuals \backslash and $/$. They are the 0-free reducts of FL-algebras, and form a variety **RL**. The logic corresponding to the whole variety **RL** is the 0-free fragment of Lambek's calculus \mathcal{FL} . For each subvariety $\mathbf{K} \subseteq \mathbf{RL}$ there is a *substructural logic*, the (finitary) *algebraizable* logic having \mathbf{K} as its *equivalent algebraic semantics*, with *defining equation* $x \wedge 1 \approx 1$ and *equivalence formula* $(x \backslash y) \cdot (y \backslash x)$. This logic will be denoted here by $\vdash_{\mathbf{K}}^1$; it is uniquely determined by the above mentioned conditions, so that it can be defined from \mathbf{K} by (3) plus the expressions

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^1 \varphi \iff \mathbf{K} \models 1 \preceq \varphi_1 \ \& \ \dots \ \& \ 1 \preceq \varphi_n \rightarrow 1 \preceq \varphi, \quad (12)$$

$$\emptyset \vdash_{\mathbf{K}}^1 \varphi \iff \mathbf{K} \models 1 \preceq \varphi, \quad (13)$$

where the symbol \models represents first-order (or quasi-equational) validity, $\&$ and \rightarrow represent first-order conjunction and implication, and \preceq represents formal order: an evaluation $v \in \text{Hom}(\mathbf{Fm}, \mathbf{A})$ satisfies an expression $\varphi \preceq \psi$ if and only if $v(\varphi) \leq v(\psi)$. Since the order has a lattice structure, actually this formal order is equivalent to each one of the equations $\varphi \wedge \psi \approx \varphi$ and $\varphi \vee \psi \approx \psi$; thus, the defining equation can be written $1 \preceq x$, as we have done in (12) and (13).

It is clear that $\vdash_{\mathbf{K}}^1$ is the logic preserving truth with respect to the class of truth structures $\{\langle \mathbf{A}, [1] \rangle : \mathbf{A} \in \mathbf{K}\}$, where the truth set

is $[1] = \{a \in A : 1 \leq a\}$. While (12) expresses that the consequence preserves truth with respect to this truth set in all algebras in \mathbf{K} , (13) expresses the fact that the set $[1]$ represents (absolute) truth in each structure, with 1 being “the least truth” in the structure.

I would like to present an alternative way of implementig (11) which still satisfies (i) and (iii), and does not suffer from the problem outlined in (iv), at the cost of abandoning (ii). Still, as I think the resulting logics fall under (11), they can legitimately be called *substructural*. This general scheme is given by the above discussed notion of a logic preserving degrees of truth (4). The problem to be solved is obviously how to select the truth degrees to be considered in each algebra in \mathbf{K} .

A potential solution might be inspired in the truth structures just considered to define $\vdash_{\mathbf{K}}^1$. While in general there need not be a connection between truth values (all elements of the algebra) and truth degrees (the subsets T_i), it is however a natural choice, much in line with (5), to associate with each truth value $a \in A$ the truth degree “having a truth value greater than or equal to a ”, represented by the set $[a] = \{b \in A : a \leq b\}$. Thus, the logic preserving degrees of truth with respect to the class $\{\langle \mathbf{A}, \{[a] : a \in A\} \rangle : \mathbf{A} \in \mathbf{K}\}$, denoted by $\vdash_{\mathbf{K}}^{\leq}$, would then be defined, according to (4), by (3) and

$$\begin{aligned} \Gamma \vdash_{\mathbf{K}}^{\leq} \varphi &\iff \forall \mathbf{A} \in \mathbf{K}, \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \forall a \in A, \\ &\text{if } a \leq v(\gamma) \forall \gamma \in \Gamma \text{ then } a \leq v(\varphi), \end{aligned} \quad (14)$$

for all (finite) $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$. Since preserving all truth sets $[a]$ in particular implies preserving $[1]$, clearly $\vdash_{\mathbf{K}}^1$ is an *extension* of $\vdash_{\mathbf{K}}^{\leq}$. There are closer relationships between the two logics, as I show below. This proposal is not without problems, though, as I will show when considering the case $\Gamma = \emptyset$. However, let us first consider the case of a non-empty $\Gamma = \{\varphi_1, \dots, \varphi_n\}$; using elementary properties of lattices it is easy to see that (14) is equivalent to:

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^{\leq} \varphi &\iff \forall \mathbf{A} \in \mathbf{K}, \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \\ &v(\varphi_1) \wedge \dots \wedge v(\varphi_n) \leq v(\varphi), \\ &\iff \mathbf{K} \models \varphi_1 \wedge \dots \wedge \varphi_n \preceq \varphi. \end{aligned} \quad (15)$$

Since by definition the algebras in \mathbf{K} have a semilattice reduct, (15) tells us that $\vdash_{\mathbf{K}}^{\leq}$ is a *semilattice based logic* with respect to \mathbf{K} in the sense

of the very general scheme of defining logics from semilattice ordered algebras first introduced in [8] and christened and studied in depth within abstract algebraic logic in [12].

From (15) several significant consequences follow. First,

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^{\leq} \varphi \iff \varphi_1 \wedge \dots \wedge \varphi_n \vdash_{\mathbf{K}}^{\leq} \varphi. \quad (16)$$

Algebraically, this says that $\vdash_{\mathbf{K}}^{\leq}$ is a **conjunctive** logic, that is, that the connective \wedge behaves as a logical (or extensional) conjunction. We see that, regarding consequence, this connective translates the metalogical punctuation sign “,” the conjoining of assumptions. I want to insist in the remarkable fact that this does not appear as a consequence of a previous metalogical choice in the process of defining the logic from the algebras, since only the order relation appears in (14), but as a consequence of the algebraic structure of the algebras (they are all lattices).

Second, using that in a residuated lattice $a \leq b \iff 1 \leq a \setminus b$, the main relation between $\vdash_{\mathbf{K}}^{\leq}$ and $\vdash_{\mathbf{K}}^1$ results from (15):

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^{\leq} \varphi \iff \emptyset \vdash_{\mathbf{K}}^1 (\varphi_1 \wedge \dots \wedge \varphi_n) \setminus \varphi. \quad (17)$$

It is remarkable that this coincides with WÓJCICKI’s proposal [19, pp. 163–170] of a solution to what he calls *the deduction problem*, that of defining a consequence from a given logical system (set of theorems) in a way that philosophically makes sense. He first considers this problem in the context of relevance logics, and his proposal was applied to the relevance system \mathcal{R} in [10], where the resulting logic was studied in depth in the context of abstract algebraic logic. Relevance logic is an important substructural logic, thus WÓJCICKI’s proposal is actually a particular case of the general idea of preservation of degrees of truth.

From (15) it seems that the logic $\vdash_{\mathbf{K}}^{\leq}$ is determined by the equational theory of \mathbf{K} . However this is only true for the consequences of non-empty sets of assumptions, while *theorems* pose a challenge. Putting $\Gamma = \emptyset$ in (14), or particularizing (6) to the present setting, we obtain

$$\emptyset \vdash_{\mathbf{K}}^{\leq} \varphi \iff \forall \mathbf{A} \in \mathbf{K}, \forall v \in \text{Hom}(\mathbf{Fm}, \mathbf{A}), \forall a \in A, a \leq v(\varphi). \quad (18)$$

Thus, the theorems of $\vdash_{\mathbf{K}}^{\leq}$ are the formulas that are always evaluated in $\bigcap_{a \in A} [a]$ for all $\mathbf{A} \in \mathbf{K}$. If the algebras in \mathbf{K} do not have a formula-definable upper bound, then the logic $\vdash_{\mathbf{K}}^{\leq}$ would have no theorems,

as happens in [10]. This is in conflict with the idea that the truth set $[1]$ represents “absolute” truth in these structures, and that theorems should be the formulas that are always interpreted as absolute truths.

Of course, this difficulty disappears when the algebras in \mathbf{K} have a formula-definable upper bound. A residuated lattice is *integral* when 1 , the unit of the monoid, is at the same time the upper bound of the lattice. In these algebras, the absolute truth set is $\bigcap_{a \in A} [a] = [1] = \{1\}$, thus representing *absolute truth* in a more classical way; the defining equation is then $x \approx 1$ and (12,13) take on a more familiar look:

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^1 \varphi \iff \mathbf{K} \models \varphi_1 \approx 1 \ \& \ \dots \ \& \ \varphi_n \approx 1 \rightarrow \varphi \approx 1 \quad (19)$$

$$\emptyset \vdash_{\mathbf{K}}^1 \varphi \iff \mathbf{K} \models \varphi \approx 1 \quad (20)$$

In this case, (14) for $\Gamma = \emptyset$ yields

$$\emptyset \vdash_{\mathbf{K}}^{\leq} \varphi \iff \mathbf{K} \models 1 \preceq \varphi \quad (21)$$

$$\iff \mathbf{K} \models \varphi \approx 1. \quad (22)$$

This has a number of pleasant consequences: The two logics have the same theorems (which seems to be a reasonable demand), $\vdash_{\mathbf{K}}^1$ is a purely inferential extension of $\vdash_{\mathbf{K}}^{\leq}$, and $\vdash_{\mathbf{K}}^{\leq}$ is completely determined by the equational theory of \mathbf{K} . This last fact has a practical advantage: in the cases where \mathbf{K} is the variety generated by a single algebra or by a well determined particular kind of its members (such as its chains, which happens often for instance in the subclass of fuzzy logics) it is enough to work with these in order to check validity of consequences.

A residuated lattice is *commutative* when its monoidal reduct is so, that is, when the operation \cdot is commutative. This property does not affect algebraizability, it just simplifies presentation: the two residuals collapse into one, commonly denoted as \rightarrow (not to be confused with \Rightarrow). The paper [2] contains a thorough study, mainly with abstract algebraic logic techniques, of the logics $\vdash_{\mathbf{K}}^{\leq}$ for varieties \mathbf{K} of *integral, commutative residuated lattices*, generalizing [7], where the logic preserving degrees of truth from the variety of MV-algebras is considered. Unfortunately, there is no space here to comment on the main results.

In the more general case a parallel study has not yet been undertaken. Let me finish by outlining the difficulty: in general, if the constant 1

keeps the signification of the “least absolute truth” and the truth set [1] that of “absolute degree of truth”, then (18) does not seem to be a reasonable definition of theorems of a logic preserving degrees of truth: the role of “logical upper bound” should be played by the set [1] rather than by the algebraic upper bound that (18) requires. But since this is a straight consequence of (14) for $\Gamma = \emptyset$, perhaps it is *the choice of degrees of truth* associated with each $\mathbf{A} \in \mathbf{K}$ that should be rethought again. It is clear that at least the degrees $[a]$ for $a > 1$ should be excluded, since for them $[a] \not\subseteq [1]$ and it does not make sense to have degrees of truth more strict than one interpreted as “absolute truth” as [1]. But not only these. In (6) we saw that the family $\{T_i : i \in I\}$ of truth degrees should be such that $\bigcap_{i \in I} T_i$ represents absolute truth in the structure. If we want to stick to truth degrees of the form $[a]$ then the structure to be considered is exactly $\langle \mathbf{A}, \{[a] : a \in A, a \leq 1\} \rangle$; however I consider that such an *ad hoc* choice is still in need of a sounder motivation. Formally we get (21), which matches well with (13), and the two logics have the same theorems; but it is not difficult to show that instead of (15) we obtain

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{K}}^{\leq} \varphi \iff \mathbf{K} \models \varphi_1 \wedge \dots \wedge \varphi_n \wedge 1 \preceq \varphi. \quad (23)$$

As a consequence, $\vdash_{\mathbf{K}}^{\leq}$ is no longer a semilattice based logic in the sense of [12], and the fundamental results of this paper cannot be used.

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References

- [1] BLOK, W., AND PIGOZZI, D. *Algebraizable logics*, vol. 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.
- [2] BOU, F., ESTEVA, F., FONT, J. M., GIL, A., GODO, L., TORRENS, A., AND VERDÚ, V. Logics preserving degrees of truth from varieties of residuated lattices. Manuscript, 2007.

- [3] CZELAKOWSKI, J. *Protoalgebraic logics*, vol. 10 of *Trends in Logic - Studia Logica Library*. Kluwer, Dordrecht, 2001.
- [4] DOŠEN, K. Historical introduction to substructural logics. In [5], pp. 1–30.
- [5] DOŠEN, K., AND SCHROEDER-HEISTER, P., Eds. *Substructural Logics*, vol. 2 of *Studies in Logic and Computation*. Oxford Science Publications, Oxford, 1993.
- [6] FONT, J. M. Generalized matrices in abstract algebraic logic. In *Trends in Logic. 50 years of Studia Logica*, V. F. Hendriks and J. Malinowski, Eds., vol. 21 of *Trends in Logic - Studia Logica Library*. Kluwer, Dordrecht, 2003, pp. 57–86.
- [7] FONT, J. M., GIL, A., TORRENS, A., AND VERDÚ, V. On the infinite-valued Łukasiewicz logic that preserves degrees of truth. *Archive for Mathematical Logic* 45 (2006), 839–868.
- [8] FONT, J. M., AND JANSANA, R. *A general algebraic semantics for sentential logics*, vol. 7 of *Lecture Notes in Logic*. Springer-Verlag, 1996. 135 pp. Distributed by the Association for Symbolic Logic.
- [9] FONT, J. M., JANSANA, R., AND PIGOZZI, D. A survey of abstract algebraic logic. *Studia Logica (Special issue on Abstract Algebraic Logic, Part II)* 74 (2003), 13–97.
- [10] FONT, J. M., AND RODRÍGUEZ, G. Algebraic study of two deductive systems of relevance logic. *Notre Dame Journal of Formal Logic* 35 (1994), 369–397.
- [11] GALATOS, N., JIPSEN, P., KOWALSKI, T., AND ONO, H. *Residuated lattices: an algebraic glimpse at substructural logics*, vol. 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.
- [12] JANSANA, R. Selfextensional logics with a conjunction. *Studia Logica* 84 (2006), 63–104.
- [13] ŁOŚ, J. AND SUSZKO, R. Remarks on sentential logics. *Indagationes Mathematicae* 20 (1958), 177–183.
- [14] NOWAK, M. Logics preserving degrees of truth. *Studia Logica* 49 (1990), 483–499.
- [15] ONO, H. Semantics for substructural logics. In [5], pp. 259–291.

- [16] ONO, H. Substructural logics and residuated lattices - an introduction. In *Trends in Logic. 50 years of Studia Logica*, V. F. Hendriks and J. Malinowski, Eds., vol. 21 of *Trends in Logic - Studia Logica Library*. Kluwer, Dordrecht, 2003, pp. 193–228.
- [17] RAFTERY, J. Correspondences between Gentzen and Hilbert systems. *The Journal of Symbolic Logic* 71 (2006), 903–957.
- [18] TARSKI, A. Über einige fundamentale Begriffe der Metamathematik. *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III, 23* (1930), 22–29.
- [19] WÓJCICKI, R. *Theory of logical calculi. Basic theory of consequence operations*, vol. 199 of *Synthese Library*. Reidel, Dordrecht, 1988.

JOSEP MARIA FONT

Department of Probability, Logic and Statistics
Faculty of Mathematics, University of Barcelona
jmfont@ub.edu