An Abstract Algebraic Logic view of some multiple-valued logics

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Abstract

Abstract Algebraic Logic is a general theory of the algebraization of deductive systems arising as an abstraction of the well-known Lindenbaum-Tarski process. The notions of logical matrix and of Leibniz congruence are among its main building blocks. Its most successful part has been developed mainly by BLOK, PIGOZZI and CZELAKOWSKI, and obtains a deep theory and very nice and powerful results for the so-called protoalgebraic logics. I will show how the idea (already explored by WÓJCKICI and NOWAK) of defining logics using a scheme of "preservation of degrees of truth" (as opposed to the more usual one of "preservation of truth") characterizes a wide class of logics which are not necessarily protoalgebraic and provide another fairly general framework where recent methods in Abstract Algebraic Logic (developed mainly by JANSANA and myself) can give some interesting results. After the general theory is explained, I apply it to an infinite family of logics defined in this way from subalgebras of the real unit interval taken as an MV-algebra. The general theory determines the algebraic counterpart of each of these logics without having to perform any computations for each particular case, and proves some interesting properties common to all of them. Moreover, in the finite case the logics so obtained are protoalgebraic, which implies they have a "strong version" defined from their Leibniz filters; again, the general theory helps in showing that it is the logic defined from the same subalgebra by the truth-preserving scheme, that is, the corresponding finite-valued logic in the most usual sense. However, for infinite subalgebras the obtained logic turns out to be the same for all such subalgebras and is not protoalgebraic, thus the ordinary methods do not apply. After introducing some (new) more general abstract notions for non-protoalgebraic logics I can finally show that this logic too has a strong version, and that it coincides with the ordinary infinite-valued logic of Łukasiewicz.

1 On Abstract Algebraic Logic

In papers on the algebraic study of a specific logic it is common to read sentences like "[such and such class of algebras] plays in relation to [such and such logic] a role similar to that played by Boolean algebras in relation to classical logic". Many works in the Algebraic Logic literature are devoted to the study of particular logics and the particular associated class of algebras, and often the said "role" amounts to very little more than the completeness theorem. Thus most of the benefits of having an algebraic counterpart of a logic were usually obtained by suitably devised ad-hoc procedures. ***Abstract Algebraic Logic*** is the branch or part of Algebraic Logic where the emphasis is put on the process of algebraization itself rather than on its results for this or that logic, and where the process is analysed and described at a truly *abstract* level. The *general* theories it develops account for the algebraization of particular logics, and can be used to obtain properties of the logics from those of the algebras or vice-versa, once their connection has been established. It also identifies, either by metalogical or by algebraic conditions, some classes of logics where certain methods can be used with particular success, the connection logic-algebras acquiring varying degrees of intensity.

One of the distinctive features of Abstract Algebraic Logic is the very definition of what a *logic* is; not taking this into account may lead to some misunderstandings. TARSKI's conception of logic as *consequence* is adopted: A ***logic*** or ***deductive system*** $\mathcal{S}$ is here identified with a finitary and substitution-invariant consequence relation on the set Fm of formulas; that is, a binary relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm) \times Fm$ such that:

1. $\Gamma \vdash_{\mathcal{S}} \varphi$ whenever $\varphi \in \Gamma$.

2. $\Gamma \vdash_{\mathcal{S}} \varphi$ whenever $\Delta \vdash_{\mathcal{S}} \varphi$ and $\Delta \subseteq \Gamma$.

3. $\Gamma \vdash_{\mathcal{S}} \varphi$ whenever $\Delta \vdash_{\mathcal{S}} \varphi$ and $\Gamma \vdash_{\mathcal{S}} \beta$ for every $\beta \in \Delta$.

4. $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\sigma\Gamma \vdash_{\mathcal{S}} \sigma\varphi$ for every substitution σ.

5. $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma_0 \vdash_{\mathcal{S}} \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.

For very general theoretical studies it may be useful to drop the finitarity condition 5, as is done in [11, 15, 44]; in such cases it is better to speak of ***consequence relations***. Note that Abstract Algebraic Logic departs from the approaches where a logic is identified with a *set of formulas* closed under some conditions or rules. It also departs from those where a logic is understood as a *proof system* of a certain kind, and from those requiring that each logic should always have both a semantics and a proof theory. While any of these devices can *define* a logic, they are not regarded as part of the notion of logic itself, but as *properties* a logic may or may not have.

The second distinctive feature of Abstract Algebraic Logic (or of Algebraic Logic in general) is that its models are taken on *algebras* of the same similarity type as the language of the formulas, and that the interpretations or evaluations are the *homomorphisms* from the formula algebra $\boldsymbol{Fm}$ to the algebra where the model resides. The model itself can be some kind of structure over the algebra, such as a subset or a family of subsets.

The most classical and best developed part of Abstract Algebraic Logic uses *subsets* as models; its central notions are those of logical matrix and of Leibniz operator. It is well-known that the origin of the notion of logical matrix can be traced back to the twenties, or even before. The general theory of matrix semantics was established through the work of WÓJCICKI [42], CZELAKOWSKI [9], RASIOWA [34] and many others, and completed its maturity after the introduction of the notion of Leibniz operator by BLOK and PIGOZZI [2] and its systematic study by themselves and other people. A (logical) ***matrix*** is a pair $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra of suitable similarity type, and $F \subseteq A$ is the set of so-called *designated elements,* which represent *truth* in the model. One says that $\langle \boldsymbol{A}, F\rangle$ is a ***matrix for*** $\mathcal{S}$ (briefly, an $\mathcal{S}$***-matrix***) when for every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$ the following holds:

$$\text{For every } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \text{ if } v[\Gamma] \subseteq F \text{ then } v(\varphi) \in F. \tag{1}$$

The set F is then called an $\mathcal{S}$***-filter***. For each algebra $\boldsymbol{A}$ the family of all the $\mathcal{S}$-filters on $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

The ***Leibniz congruence*** of a matrix $\langle \boldsymbol{A}, F\rangle$ is defined as

$$\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \max\{\theta \in \mathrm{Co}\boldsymbol{A} : \text{ if } \langle a, b\rangle \in \theta \text{ and } a \in F \text{ then } b \in F\},$$

where $\mathrm{Co}\boldsymbol{A}$ denotes the set of (algebraic) congruences of the algebra $\boldsymbol{A}$. A matrix is ***reduced*** when its Leibniz congruence is the identity. The first class of algebras naturally associated with a logic, denoted by $\mathsf{Alg}^*\mathcal{S}$, is the class of the algebraic reducts of the reduced matrices of $\mathcal{S}$, that is, the class of algebras $\boldsymbol{A}$ such that there is $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ with $\langle \boldsymbol{A}, F\rangle$ reduced; notice that this F need not be unique. It sometimes happens that the Leibniz congruence and the reduced matrices of a given logic can be nicely represented; for instance in the implicative logics studied in [34] $\langle a, b\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ if and only if $a \to b, b \to a \in F$, and the algebras in $\mathsf{Alg}^*\mathcal{S}$, there called "$\mathcal{S}$-algebras", are determined by the axioms and rules of the logic plus the familiar condition "if $a \to b = b \to a = 1$ then $a = b$". But in less well-behaved cases things can be much different.

The historical development of Abstract Algebraic Logic can be identified with the process of extending some paradigms of the algebraization of logic, which had proven successful in the study of the best-behaved logics, to wider and wider classes of logics; the extension has been performed in a way that obtains deep and meaningful results and as powerful and nice a theory as possible, while keeping the old results in the already studied cases. The generalization of the Lindenbaum-Tarski process to ***implicative*** logics [34] has been extended and specialised for other (increasingly larger) classes of logics: the ***algebraizable*** ones [3, 12], the ***equivalential*** ones [10], and the ***protoalgebraic*** ones [2, 4]. Each of these three classes of logics can be characterized by a certain aspect of the behaviour of ***the Leibniz operator***: the mapping $\boldsymbol{\Omega}_{\boldsymbol{A}} : F \mapsto \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$ when F ranges over $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The result has been a powerful, complex and multifaceted theory, which by now forms the established core of Abstract Algebraic Logic, as developed in [4, 5, 11]; [17] is a compact, yet comprehensive survey of recent work in the area. I am just going to give the definitions and properties I will use in the paper.

The classes of logics mentioned in the previous paragraph, together with a few others, form the so-called ***hierarchy*** of logics, also called the *protoalgebraic hierarchy*, the *algebraic hierarchy* or the *Leibniz hierarchy*. A logic $\mathcal{S}$ is ***protoalgebraic*** when for every $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ is monotonic on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, that is, $F \subseteq G$ implies $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$ for all $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. $\mathcal{S}$ is ***equivalential*** when there is a set $\Delta(p, q)$ of formulas in two variables that defines the Leibniz congruence on $\mathcal{S}$-filters in the following sense: For any $\boldsymbol{A}$, if $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a, b \in A$, then

$$\langle a, b\rangle \in \boldsymbol{\Omega}_{\boldsymbol{A}}(F) \iff \Delta^{\boldsymbol{A}}(a, b) \subseteq F. \tag{2}$$

The set $\varDelta$ is the set of ***equivalence formulas*** of $\mathcal{S}$. When this set can be taken finite then $\mathcal{S}$ is called ***finitely equivalential***. $\mathcal{S}$ is ***weakly algebraizable*** when the Leibniz operator in injective on $\mathcal{S}$-filters; it is ***algebraizable*** when it is both equivalential and weakly algebraizable. The original definition of this notion by BLOK and PIGOZZI in [3] is now called ***finitely algebraizable***, and corresponds to the logics that are both weakly algebraizable (or algebraizable) and finitely equivalential; for these logics the class $\mathsf{Alg}^*\mathcal{S}$ is a quasivariety, which is called ***the equivalent algebraic semantics*** for $\mathcal{S}$. The links between the logic and this class of algebras are very strong, for instance the Leibniz operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ becomes an *isomorphism* between the lattices $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Co}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ (the set of congruences of $\boldsymbol{A}$ yielding a quotient in $\mathsf{Alg}^*\mathcal{S}$). Further kinds of algebraizability are denoted by an additional adjective to each of the just mentioned classes: ***regularly*** means that each algebra $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}$ has a special element 1 such that $\mathcal{S}$ is complete with respect to the class of matrices $\big\{\big\langle \boldsymbol{A}, \{1\}\big\rangle : \boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}\big\}$; **strongly** means that the class $\mathsf{Alg}^*\mathcal{S}$ is a *variety*. Figure 1 shows the organisation of the main classes of the hierarchy appearing in this paper.

implicative logics		
↓		
finitely regularly algebraizable logics		
↓		
finitely algebraizable logics	→	finitely equivalential logics
↓		↓
algebraizable logics	⟶	equivalential logics
↓		↓
weakly algebraizable logics	→	**protoalgebraic logics**

Figure 1: Some of the main classes of logics in the hierarchy. $\rightarrow$ means $\subseteq$.

Protoalgebraic logics, the largest class in the hierarchy, include the vast majority of logics usually considered in the literature (classical and intuitionistic logics, modal logics, many-valued logics, etc.). They are considered to be the largest class of logics to which the standard model-theoretic methods of the theory of logical matrices can be successfully applied, beyond the most general completeness theorems. The following characterization is of a special interest: A logic is protoalgebraic if and only if there is a set $E(p\,,q)$ of formulas in two variables satisfying the following two (minimal) requirements, for all formulas $\varphi\,,\psi$:

$$\text{(Law of Identity)} \qquad \vdash_{\mathcal{S}} E(\varphi\,,\varphi)$$
$$\text{(Modus Ponens)} \qquad E(\varphi\,,\psi) \cup \{\varphi\} \vdash_{\mathcal{S}} \psi$$

From this it results that non-protoalgebraic logics must have no implication at all, or at most a rather strange one; formerly it was believed that only very pathological logics would be non-protoalgebraic, but recently their interest has been recognized, in parallel to the identification of several families of (natural) examples: the conjunction-disjunction and the implication-less fragments of intuitionistic logic [20, 35], some subintuitionistic logics [1, 7, 36, 39, 41], BELNAP's four-valued logic [13], and the weak version of system $\mathcal{R}$ of relevance logic, defined by following WÓJCICKI's suggestions in [44, p. 165] and algebraically studied in [19]. And, as I will show in the final section, there is also an infinite multiple-valued logic in this group. The algebraic treatment of these logics clearly calls for another framework with a wider scope.

This recent branch of Abstract Algebraic Logic has grown around the notions of generalized matrix [42], of Tarski congruence, and of full model. A ***generalized matrix*** (called ***abstract logic*** in [6] and in [14]) is a pair $\langle \boldsymbol{A}, \mathcal{C}\rangle$ where $\mathcal{C}$ is a closed-set system (i.e., a family of subsets closed under arbitrary intersections and containing the whole universe) on an algebra $\boldsymbol{A}$. It is a ***model of a logic*** $\mathcal{S}$ when for every $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ then $v(\varphi) \in C(v[\Gamma])$, where C is the closure operator associated with the closed-set system $\mathcal{C}$. Obviously, $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a model of $\mathcal{S}$ if and only if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; thus on any algebra, an arbitrary collection of $\mathcal{S}$-filters constitutes a model. This means that not much can be said about models in general, but some can be selected as behaving in more interesting ways. Observe that on any algebra there is a "largest" model $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle$; and it turns out that models that are "like" these are seen to have more interest. Models of this kind are called ***basic full models***, and a generalized matrix is a ***full model of*** $\mathcal{S}$ when it is the inverse image of a basic full model of $\mathcal{S}$ under a strict surjective homomorphism between generalized matrices; a surjective $h \in \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A})$ is ***strict*** between $\langle \boldsymbol{B}, \mathcal{D}\rangle$ and $\langle \boldsymbol{A}, \mathcal{C}\rangle$ when $\mathcal{D} = h^{-1}[\mathcal{C}]$. The ***Tarski congruence*** of a generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is defined as

$$\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathcal{C}) = \bigcap \big\{ \boldsymbol{\Omega}_{\boldsymbol{A}}(F) : F \in \mathcal{C} \big\}$$

and $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is ***reduced*** when its Tarski congruence is the identity. Then the class of $\mathcal{S}$***-algebras***, the second class of algebras canonically associated with a logic, is defined as the class of algebraic reducts of reduced models of $\mathcal{S}$; it is denoted by $\mathsf{Alg}\mathcal{S}$, and it happens to coincide with the class of algebraic reducts of reduced full models of $\mathcal{S}$. The study of these notions has been developed mainly in [14], where the thesis is maintained (by developing a consistent general theory and by analysing many examples) that they account for the algebraization of arbitrary logics in a faithful and meaningful way. In particular, it seems that $\mathsf{Alg}\mathcal{S}$ deserves the title of ***the algebraic counterpart of a logic*** much better than $\mathsf{Alg}^*\mathcal{S}$, specially in the non-protoalgebraic cases; moreover, if $\mathcal{S}$ is protoalgebraic then $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$, hence in the best-behaved and well-known cases the classical theory is recovered.

In the quest for new and more encompassing general theories the *study of examples*, either taken individually or in groups that share certain features, is essential. They are needed to test the theory against practice, to confirm (or, in some cases, surprisingly reject) the intuitive ideas or the results obtained by ad-hoc constructions not conforming to any precise methodology, to compare with existing paradigms, to help identify and sort the key notions from the peripheral ones, etc.

In this paper I try to show how the application of a certain general framework, developed in [16] in the context just introduced, helps in understanding and describing the algebraic behaviour of a certain family of multiple-valued logics defined from truth-value algebras that are subalgebras of the real unit interval endowed with Łukasiewicz's familiar operations. The way these logics are defined from the algebras is a particular instance of the so-called ***semilattice-based logics****, a general procedure devised in order to formalise a specific view of logics as

*The term "(semi)lattice-based" has been used in [38], in a non-technical way, to describe a large class of logics inference systems preserving degrees of truth. I devote Section 2 to an informal exposition of this idea, and Section 3 to summarize the main elements of the general theory of [16] that will be used in the sequel. Then Sections 4 and 5 contain the detailed treatment of Łukasiewicz-based logics.

As general references on Abstract Algebraic Logic I recommend [3, 5, 11, 14, 17, 34, 44]; for multiple-valued logics the recent survey monographs [8, 24, 25] contain a lot of information.

2 Logics preserving degrees of truth

Non-standard truth values are the basis of any rationale behind the setting up of a multiple-valued logic on semantical grounds. When a logic is defined through the use of a set of more than two values, these 'values' are presented as encoding different *forms, ways* or *kinds* of *being true*, and are called 'truth-values'; the classical 0 and 1 representing the extreme cases of *absolute truth* and *absolute falsity*. However, the way these truth-values are actually used in the definition of the logic may point to a different interpretation, and supports the study of an alternative way of using them.

The most common framework for definition of multiple-valued logics is to start from some *set of truth-values* A and among them to select or *designate* a certain element $1 \in A$ as representing *truth*. Then, given some set Val of *evaluations*, that is, functions $Fm \to A$, one can define a consequence relation $\vdash^1$ in the following way: For any $\Gamma \subseteq Fm$ and any $\varphi \in Fm$,

$$\Gamma \vdash^1 \varphi \iff \text{for all } v \in Val : v(\varphi) = 1 \text{ whenever for all } \beta \in \Gamma,\ v(\beta) = 1\,. \tag{3}$$

Logics defined in this way are usually said to follow a ***truth-preserving*** scheme. However, if one wants to really believe that all elements of A represent some kind of truth, of which 1 represents absolute truth, then (3) should be rather regarded as a "preservation of absolute truth" scheme, as it does not guarantee the preservation of any other truth-value than absolute truth: the elements of A are used as possible values for the computation of the value $v(\varphi)$ of non-atomic formulas φ from the values of their atomic parts (variables), but then only those evaluations giving final truth-value 1 to the formulas are taken into account in order to define consequence. This way of *using* the truth-values induces one to think that the other values do not carry any kind of truth in themselves, and that they rather represent "kinds of falsity", or simply "values" which are not "truth-values".

The same comment can be made in case one selects or designates a subset $D \subseteq A$ instead of a single element: If the definition is

$$\Gamma \vdash^D \varphi \iff \text{for all } v \in Val : v(\varphi) \in D \text{ if for all } \beta \in \Gamma,\ v(\beta) \in D\,. \tag{4}$$

then only final values inside D count, and the "truth-content" of values outside D seems not to be relevant for the logic so defined.

An alternative way of using a set A of 'truth-values' as truly representing different kinds of truth, is to think of them as *degrees of truth*, and to *understand consequence in the following sense*: That whenever all premisses attain at least a certain degree of truth, the conclusion should

having both algebraic and relational semantics linked by representation theorems of different kinds.

have at least that degree of truth too. This means assuming that there is some (partial) *ordering relation* $\leq$ among the elements of A, and defining a consequence relation $\vdash^{\leq}$ in the following way:

$$\Gamma \vdash^{\leq} \varphi \iff \text{For all } v \in Val \text{ and all } t \in A \ : \ v(\varphi) \geq t \ \text{ if for all } \beta \in \Gamma\,,\ v(\beta) \geq t\,. \tag{5}$$

This scheme is referred to as ***preservation of degrees of truth***.

In the case where one wants the logic to be truth-functional, one assumes that the set of truth-values A has an algebraic structure $\boldsymbol{A} = \langle A, \{\lambda^{\boldsymbol{A}} : \lambda \in \mathcal{L}\}\rangle$ of the similarity type $\mathcal{L}$ of the formulas, and takes $Val = \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ as the set of evaluations. Then, both definition schemes can be smoothly represented in a more algebraic-logic style by means of *matrices*, in the technical sense described in Section 1: Schemes (3) and (4) correspond to the logic defined by the logical matrices

$$\big\langle \boldsymbol{A}, \{1\} \big\rangle \qquad \text{or} \qquad \big\langle \boldsymbol{A}, D \big\rangle \tag{6}$$

respectively, while (5) corresponds to the logic defined by the family of matrices

$$\big\{ \big\langle \boldsymbol{A}, [t) \big\rangle : t \in A \big\} \tag{7}$$

where $[t) = \{a \in A : t \leq a\}$. Equivalently, (5) corresponds to the logic defined by the generalized matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ where $\mathcal{C}$ is the closed-set system generated by the family of sets $\big\{[t) : t \in A\big\}$.

These definitions might seem too algebraic, and be considered too restrictive as general schemes for defining logics. However, WÓJCICKI has shown in [44, Chapter 5] that any logic defined either locally or globally by a relational semantics of the most general kind, and hence apparently non-truth-functional, can also be defined by a class of matrices or of generalized matrices; therefore, it happens to be truth-functional with respect to convenient algebraic structures, which are obtained through suitable representation constructions from the relational structures. This issue has also been reviewed, for a large class of particular cases, in [38].

The idea of preservation of the degree of truth is not new. It has been surely discussed in a variety of works related to multiple-valued logics or even more in general, in connection with matrix semantics, as in [44, p. 191]†. Its specific algebraic side has been less studied, though. It is applied to finite subalgebras of the real unit interval in [21, 44] (see Section 4 for details) and it is studied in general by NOWAK in [33]; in NOWAK's paper the set of truth-values is supposed to have (or to be embedded in) a complete lattice structure with maximum 1, hence (5) can be rephrased as

$$\Gamma \vdash^{\leq} \varphi \iff v(\varphi) \geq \inf\big(\{v(\beta) : \beta \in \Gamma\} \cup \{1\}\big)\ , \ \text{for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}). \tag{8}$$

Notice that this is just one of the several possible notions of "preserving degrees of truth" that are introduced and characterised in [33]. However, it turns out that the consequence relations defined in any of the preceding ways cannot in general be guaranteed to be *finitary*, unless the set A is finite, see Theorem 20. I am going to show that, by restricting (3) and (5) to finite Γ,

†Warning: in this book the term "truth-preserving" is used in the sense explained above, except on page 345 where it means "preserving degrees of truth" (and at that place "preserving validity" is used as a replacement for "truth-preserving").

one obtains a reasonably smooth general framework to be exploited with the tools of Abstract Algebraic Logic, and which includes many usual logics, while it does not require them to be even protoalgebraic. Moreover, it is not necessary to assume a complete lattice structure, as in [33]; just assuming an *inf-semilattice with maximum* is enough. After these two changes, if as usual the semilattice operation is denoted by $\wedge$ and its maximum is denoted by 1, then condition (8) can be split into the following two:

$$\varphi_0, \dots, \varphi_{n-1} \vdash^{\leq} \psi \iff v(\varphi_0) \wedge \cdots \wedge v(\varphi_{n-1}) \leq v(\psi) \text{ for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \qquad (8')$$

$$\vdash^{\leq} \psi \iff v(\psi) = 1 \text{ for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \qquad (8'')$$

One typical feature of these schemes, which becomes even clearer now, is that the theorems of the logic $\vdash^{\leq}$ will be the same as those of the logic $\vdash^{1}$. It is clear that any possible interest of the present proposal must lie in the *inferential* aspect of logics, rather than in their *assertional* aspect. The problem of how to associate a consequence, or *entailment relation* with a given set of "theorems" or "tautologies" is surely a non-trivial one, and has been discussed many times and from many points of view. I want to highlight here the discussion in [44, Section 2.10] because of its connection with the more specific part of the present paper: there, an operation $\to$ of "implication" is assumed to exist, which establishes a strong connection between the two key elements at work, namely the ordering relation and the maximum truth-value; applied to the present case this would become:

$$\text{For all } a, b \in A, \quad a \leq b \iff a \to b = 1. \qquad (9)$$

Any logic defined with (8') and (8") from a truth-value algebra where (9) holds, satisfies

$$\varphi_0, \dots, \varphi_{n-1} \vdash^{\leq} \psi \iff \vdash^{\leq} \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \to \psi. \qquad (10)$$

Viewed the other way round: Imagine that a set of "tautologies" has been defined from $\boldsymbol{A}$ and 1 through (8"). Then if one wants to have connectives $\wedge$ and $\to$ representing respectively conjunction and inference *inside the language* so that (10) holds, then the natural way is to go for (8') and find the logic $\vdash^{\leq}$ defined by preservation of degrees of truth. However, having (9) is a rather particular property that may not be present in many cases, for instance in all those cases where simply there is no such implication in the language, or when one precisely wants to deal with an implication-less fragment of a richer logic; in those cases (8') and (8") still offer a sound definition of a logic preserving degrees of truth.

Condition (8') or its more general form (5) are often paraphrased as stating that "the conclusion must have at least the degree of truth of the premisses". Since in logical inference premisses act collectively, it is generally acknowledged that a reasonable evaluation of the "collective" degree of truth of the set of premisses is the infimum of their degrees of truth. It is interesting to notice that a similar intuition is present in the characterizations of the notions of a fuzzy subset being a fuzzy subalgebra of a (crisp) algebra [31] and of canonical fuzzy numbers [32, 30].

3 Semilattice-based logics

The material in this section is excerpted from [15] and [16].

Let K be a class of algebras of some (arbitrary but fixed from now on) similarity type having an upper-bounded inf-semilattice reduct; this means there is a partial ordering relation $\leq$ on each $\boldsymbol{A} \in \mathsf{K}$ having a maximum 1, and a binary connective $\wedge$ (which can be either a primitive one or defined by a term in two variables) such that $a \wedge b = \inf\{a, b\}$ for all $a, b \in A$ and all $\boldsymbol{A} \in \mathsf{K}$. There is no harm in assuming that the maximum is represented as a constant $\top$ of the language. It is well-known that this situation can be expressed equationally as the satisfaction in K of the four following equations

$$x \wedge x \approx x$$
$$x \wedge y \approx y \wedge x$$
$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$$
$$x \wedge \top \approx x$$

together with the requirement that for all $a, b \in A$ (for any $\boldsymbol{A} \in \mathsf{K}$),

$$a \leq b \iff a = a \wedge b.$$

Definition 1 *Let $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ be a finitary sentential logic. It is said to be* ***semilattice-based with respect to*** K ***through*** $\wedge$ *when for any $\varphi_0, \ldots, \varphi_{n-1}, \psi \in Fm$ the following hold:*

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \psi \iff v(\varphi_0) \wedge \cdots \wedge v(\varphi_{n-1}) \leq v(\psi) \text{ for all } v \in \operatorname{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \text{ and all } \boldsymbol{A} \in \mathsf{K} \qquad (11)$$

$$\vdash_{\mathcal{S}} \psi \iff v(\psi) = 1 \text{ for all } v \in \operatorname{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \text{ and all } \boldsymbol{A} \in \mathsf{K} \qquad (12)$$

Since I am assuming that all the algebras have a maximum and it is denoted by a constant in the language, it follows from (12) that all these logics will have theorems. A slightly more general setting can be obtaining by deleting these assumptions, as is done in [16]; however for the purpose of introducing the multiple-valued cases I want to deal with, one can safely assume such properties and the exposition is somehow simplified.

Elementary properties

1. Independence from $\wedge$: If $\mathcal{S}$ is also semilattice-based with respect to another class K' through another binary term $\wedge'$ then the varieties generated by K and by K' are equal, and modulo this variety the terms $\wedge$ and $\wedge'$ are equivalent.

2. If $\mathcal{S}$ is semilattice based with respect to K then it is also so with respect to $\mathsf{V}(\mathsf{K})$, and this variety is the only variety with respect to which $\mathcal{S}$ can be semilattice-based.

3. Two formulas φ and ψ are ***interderivable*** modulo $\mathcal{S}$ (a relation I denote by $\varphi \dashv\vdash_{\mathcal{S}} \psi$ and define as "$\varphi \vdash_{\mathcal{S}} \psi$ and $\psi \vdash_{\mathcal{S}} \varphi$") if and only if the equation $\varphi \approx \psi$ is true in K.

4. The interderivability relation $\dashv\vdash_{\mathcal{S}}$ is a congruence of the formula algebra $\boldsymbol{Fm}$, and the quotient algebra $\boldsymbol{Fm}/\dashv\vdash_{\mathcal{S}}$ generates the variety $\mathsf{V}(\mathsf{K})$.

5. The term $\wedge$ is a ***conjunction*** for $\mathcal{S}$, that is, it satisfies the three Hilbert-style rules

$$\varphi \wedge \psi \vdash \varphi \quad , \quad \varphi \wedge \psi \vdash \psi \quad \text{and} \quad \varphi, \psi \vdash \varphi \wedge \psi.$$

6. The logic $\mathcal{S}$ is entirely determined from its interderivability relation $\dashv\vdash_{\mathcal{S}}$, plus condition (12) for theorems. That is (given the result in item 3 above), the equational theory of $\mathbf{K}$ completely determines the inferential part of $\mathcal{S}$.

A logic $\mathcal{S}$ is called ***selfextensional*** when its interderivability relation $\dashv\vdash_{\mathcal{S}}$ is a congruence of the formula algebra $\boldsymbol{Fm}$. The logics with this property enjoy a strong substitutivity property: If $\alpha \dashv\vdash_{\mathcal{S}} \beta$ then for every $\varphi(p) \in Fm$, $\varphi(\alpha) \dashv\vdash_{\mathcal{S}} \varphi(\beta)$. This notion was highlighted and has been studied by WÓJCICKI, see [44]. The first important result about semilattice-based logics is:

Theorem 2 *A logic $\mathcal{S}$ is semilattice-based if and only if it is selfextensional, has theorems, and has a conjunction.* ∎

This result might be considered to be implicit in [33], although under weakening of some parts and strengthening of others, as discussed in Section 2.

Examples

The preceding result characterizes by three metalogical properties the logics admitting a definition in terms of preserving degrees of truth in the semilattice case. Despite its perhaps non-standard or lesser-known phrasing, I want to emphasise that *the class of logics covered by these properties is very large*: Conjunction is a very weak and common requirement, and WÓJCICKI showed that selfextensional logics are exactly the *local consequences* defined by any possible-world or frame semantics in a very general sense of the word, see [11, Section 6.7] or [44, Chapter 5]. From this it follows, for instance, that a very large group of modal logics can be studied under this framework.

Moreover, one can prove that all fragments of classical or intuitionistic logic containing conjunction $\wedge$ belong to this group. As is shown in the next section, also a big family of multiple-valued logics belongs to it. And many logics being a strengthening or an expansion of these will also belong to the same group, for instance, all modal logics referred to above (however, notice that not every expansion of a logic in this group belongs to it, as being selfextensional is a property that is not automatically preserved under strengthenings or under expansions).

This group contains many protoalgebraic logics, such as most of the just-mentioned fragments (more precisely, all those fragments containing the implication or equivalence connectives besides conjunction), but also many that are not (and normally these are much less known). Among the non-protoalgebraic examples are all those cited in Section 1, including an infinite multiple-valued logic, according to Theorem 23.

Algebraic models

It is clear from their definition that the logics I am considering bear a special relationship to the class $\mathbf{K}$ of algebras and also to the generated variety $\mathbf{V}(\mathbf{K})$. The application of the general notions and tools of Abstract Algebraic Logic show that these relations are not just the expression of the logic's definition, but something more: They conform to the general framework of algebraization of logic put forward in [14], whose particular interest arises in the treatment of logics that are not necessarily protoalgebraic.

For any algebra $\boldsymbol{A} \in \mathbf{V}(\mathbf{K})$ I denote by $\mathcal{F}ilt(\boldsymbol{A})$ the set of all ***(semilattice) filters of*** $\boldsymbol{A}$, that is, those $F \subseteq A$ satisfying:

1. $1 \in F$.

2. If $a \in F$ and $b \in F$ then $a \wedge b \in F$.

3. If $a \in F$ and $a \leq b$ then $b \in F$.

In case $\boldsymbol{A}$ is a lattice then these are just the ordinary lattice filters of $\boldsymbol{A}$. Notice that condition 3 amounts to the converse of 2. One can show:

Theorem 3 *If $\mathcal{S}$ is semilattice-based with respect to $\mathbf{K}$ then for each $\boldsymbol{A} \in \mathbf{V}(\mathbf{K})$, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}ilt(\boldsymbol{A})$, the generalized matrix $\langle \boldsymbol{A}, \mathcal{F}ilt(\boldsymbol{A}) \rangle$ is reduced, and $\mathbf{Alg}\mathcal{S} = \mathbf{V}(\mathbf{K})$.* ∎

Hence $\mathbf{V}(\mathbf{K})$ is the class of algebras canonically associated with the logic by the abstract framework, that is, it is *the algebraic counterpart of $\mathcal{S}$*. This result, besides its practical applications for particular logics, has some theoretical significance, for it answers in the affirmative a recurrent question in Abstract Algebraic Logic; namely, it identifies a large class of logics whose algebraic counterpart is a *variety*, something in general not guaranteed by the theory: even for finitely and regularly algebraizable logics $\mathcal{S}$, the general theory establishes that the class $\mathbf{Alg}\mathcal{S}$ is a quasivariety, and not necessarily a variety.

Also the class of *full models* of $\mathcal{S}$ can be characterized with respect to $\mathbf{K}$:

Theorem 4 *Let $\mathcal{S}$ be semilattice-based with respect to $\mathbf{K}$ and let $\langle \boldsymbol{A}, \mathcal{C} \rangle$ be a generalized matrix. Then $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full model of $\mathcal{S}$ if and only if there is a strict surjective homomorphism from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto some generalized matrix $\langle \boldsymbol{B}, \mathcal{D} \rangle$ such that $\boldsymbol{B} \in \mathbf{V}(\mathbf{K})$ and $\mathcal{D} = \mathcal{F}ilt(\boldsymbol{B})$.* ∎

The protoalgebraic case: the strong version

Although the main virtue of the semilattice-based framework is its independence of protoalgebraicity, nevertheless one can take also advantage from the power of the classical theory of protoalgebraic logics as developed in [2, 11] together with some recent results from [15]; the combination of the two properties will help in answering a very natural question in the present context.

Section 2 has described a situation where two logics arise naturally, one being a strengthening of the other but both sharing the same theorems. There is a more abstract situation where a similar phenomenon is observed, namely the notion of the "strong version" of a protoalgebraic logic introduced and studied in [15].

Let $\mathcal{S}$ be a protoalgebraic logic with theorems. An $\mathcal{S}$-filter F is ***Leibniz*** when $F \subseteq G$ for all $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ with $\boldsymbol{\Omega}_{\boldsymbol{A}}(F) = \boldsymbol{\Omega}_{\boldsymbol{A}}(G)$, that is, when it is the least among all $\mathcal{S}$-filters on the same algebra having the same Leibniz congruence. The definition can be given in general, but for protoalgebraic logics, thanks to the monotonicity of the operator $\boldsymbol{\Omega}_{\boldsymbol{A}}$ on the set $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ for each algebra $\boldsymbol{A}$, one can show that Leibniz filters exist for every value of $\boldsymbol{\Omega}_{\boldsymbol{A}}$; more precisely, for every $\mathcal{S}$-filter F there is a (unique) Leibniz filter F^+ with $\boldsymbol{\Omega}_{\boldsymbol{A}}(F^+) = \boldsymbol{\Omega}_{\boldsymbol{A}}(F)$; actually F^+ can be obtained as the intersection of all filters with the same Leibniz congruence as F. A matrix is Leibniz when its filter is. Then with each logic $\mathcal{S}$ one can associate the logic $\mathcal{S}^+$ defined by the class of all Leibniz matrices of the original logic $\mathcal{S}$; this logic is called ***the strong version of*** $\mathcal{S}$, and, under certain conditions, there are strong relations between $\mathcal{S}^+$ and $\mathcal{S}$, as described in [15]; the first to be immediately seen are that $\mathcal{S}^+$ *is a strengthening of* $\mathcal{S}$ *and that these two logics have the same theorems* (because the least filter on each algebra is always Leibniz). Recall that a logic $\mathcal{S}'$ is a ***strengthening*** of a logic $\mathcal{S}$ if and only if $\vdash_{\mathcal{S}} \ \subseteq \ \vdash_{\mathcal{S}'}$ as binary relations, that is, if and only if $\Gamma \vdash_{\mathcal{S}} \psi$ implies $\Gamma \vdash_{\mathcal{S}'} \psi$ for all $\Gamma \subseteq Fm$ and all $\psi \in Fm$.

Notice that for algebraizable logics (in any of the degrees of this notion: weakly, finitely, strongly, etc.) the Leibniz operator is injective (see Section 1) therefore every $\mathcal{S}$-filter is Leibniz and $\mathcal{S}^+ = \mathcal{S}$. Hence this issue is of interest only for protoalgebraic but non-algebraizable logics. In the case where the starting logic is semilattice-based, in [16] the following facts are proved:

Proposition 5 *Let $\mathcal{S}$ be a protoalgebraic logic that is semilattice-based (with respect to some class of algebras) but not weakly algebraizable. Then $\mathcal{S}^+$ is not selfextensional.* ∎

Theorem 6 *Let $\mathcal{S}$ be a protoalgebraic logic that is semilattice-based with respect to* K. *Then its strong version $\mathcal{S}^+$ is strongly, finitely and regularly algebraizable and its equivalent algebraic semantics is* $\mathsf{V}(\mathsf{K})$. *As a consequence, $\mathcal{S}^+$ is the logic defined by the class of matrices* $\big\{\big\langle \boldsymbol{A}, \{1\}\big\rangle : \boldsymbol{A} \in \mathsf{V}(\mathsf{K})\big\}$. ∎

The abstract setting can be intuitively read as follows: In the conditions of the theorem,

if $\mathcal{S}$ is defined from K by preserving degrees of truth
then its strong version $\mathcal{S}^+$ is defined from $\mathsf{V}(\mathsf{K})$ by preserving truth.

Obviously if K is already a variety then $\mathsf{K} = \mathsf{V}(\mathsf{K})$ and $\mathcal{S}^+$ coincides with the logic defined from K by preserving truth; however this condition is a very strong one, since often one wants to start with a very small class K (very often, with a single algebra!) and still have some connection between the logics defined from K by the two preservation schemes. There is another condition which will better fit in the particular situation of later sections; here $\mathsf{Q}(\mathsf{K})$ denotes the quasivariety generated by K. Then:

Theorem 7 *Let $\mathcal{S}$ be an equivalential logic that is semilattice-based with respect to* K. *Then its strong version $\mathcal{S}^+$ coincides with the logic defined by the class of matrices* $\big\{\big\langle \boldsymbol{A}, \{1\}\big\rangle : \boldsymbol{A} \in \mathsf{K}\big\}$ *if and only if* $\mathsf{Q}(\mathsf{K}) = \mathsf{V}(\mathsf{K})$. ∎

So here there are some conditions under which the relationship between the two logics defined from K by the two multiple-valued schemes analysed in Section 2 can be described in completely abstract terms, that is, without having to refer to $\leq, \wedge, 1$ or K. Moreover, in the equivalential case the stronger version can be syntactically reduced to the weak version:

Theorem 8 *Let $\mathcal{S}$ be an equivalential logic that is semilattice-based, let $\Delta(p,q)$ be its set of* *equivalence formulas, and put* $X(p) = \Delta(p, \top)$*. Then*

$$\Gamma \vdash_{\mathcal{S}+} \varphi \Longleftrightarrow X(\Gamma) \vdash_{\mathcal{S}} \varphi \tag{13}$$

for all $\Gamma \subseteq Fm$ *and all* $\varphi \in Fm$*, where* $X(\Gamma) = \bigcup\{X(\beta) : \beta \in \Gamma\}$. ∎

One can find a variety of particular cases where the situation is naturally found. The examples of normal modal logics are paradigmatic (there the weak and the strong versions correspond to the local and the global consequences generated by a class of Kripke frames) and have been dealt with at length in [15, Section 2B]; those of quantum logics, analysed in [15, Section 2A], constitute another typical group of examples.

The remaining sections are devoted to the multiple-valued case, where similar situations occur, although with some interesting particularities.

4 Logics defined from finite subalgebras of the real unit interval

Let $[\mathbf{0},\mathbf{1}]$ be the algebra on the real unit interval, with Łukasiewicz's well-known operations:

$$\begin{aligned}
\neg x &= 1 - x \\
x \to y &= \min\{1, 1 - x + y\} \\
x \vee y &= \max\{x, y\} \\
x \wedge y &= \min\{x, y\} \\
x * y &= \max\{0, x + y - 1\} \\
x \oplus y &= \min\{1, x + y\} \\
x \leftrightarrow y &= \min\{1 - x + y, 1 - y + x\}
\end{aligned}$$

where $+$ and $-$ are the ordinary arithmetical operations; as is well-known, one can take just a small subset of them as primitive and define the remaining ones by suitable equations; see [8, Chapter 4] for instance, but this issue is not relevant here. I use the customary abbreviations p^n and $n\,p$ to denote the iterated "star" and "plus" operations respectively; that is, $p^{n+1} = p^n * p$ and $(n+1)\,p = (n\,p) \oplus p$ for $n \geq 1$, and $p^1 = 1\,p = p$.

The algebras in the variety $\mathsf{MV} = \mathsf{V}([\mathbf{0},\mathbf{1}])$ generated by the algebra $[\mathbf{0},\mathbf{1}]$ have received several names in the literature; the best-known two are ***MV-algebras*** and ***Wajsberg algebras***, the latter being used mostly when it is presented with the operations $\neg$ and $\to$ as the primitive ones. I assume that the language has a constant connective $\top$ and that $\top^{\boldsymbol{A}} = 1$ in any $\boldsymbol{A} \in \mathsf{MV}$; I also write 0 for $\neg 1$; the elements 0 and 1 are, respectively, the lower and upper bounds of their lattice structure. For their logical, algebraic and lattice-theoretical properties, and those of special subvarieties and subquasivarieties, one can read [8, 18, 23] and other papers therein referenced.

Let $\boldsymbol{S}$ be any subalgebra of $[\mathbf{0},\mathbf{1}]$; note that $0, 1 \in S$ because I have included $\top$ in the language, but this would be the case even without this, since $a \to a = 1$ and $\neg(a \to a) = 0$ for any $a \in [0,1]$. Moreover, the set S with the natural order of real numbers $\leq$ is a bounded lattice, its operations being $\wedge$ and $\vee$, and 1 its maximum. Therefore, with each such subalgebra one can associate two sentential logics following the general schemes previously discussed: the first one, denoted by $Ł_{\boldsymbol{S}}^{\leq}$, is defined by the preservation of degrees of truth schemes (11) and (12), and the second one, denoted by $Ł_{\boldsymbol{S}}$, is defined by the preservation of truth scheme (3); in both cases $\mathsf{K} = \{\boldsymbol{S}\}$. Therefore:

Definition 9 *For each subalgebra $\boldsymbol{S}$ of $[\mathbf{0}{,}\mathbf{1}]$ the logics $Ł_{\boldsymbol{S}}^{\leq} = \langle \boldsymbol{Fm}, \vdash_{\boldsymbol{S}}^{\leq} \rangle$ and $Ł_{\boldsymbol{S}} = \langle \boldsymbol{Fm}, \vdash_{\boldsymbol{S}}^{1} \rangle$ are the logics defined by the following specifications:*

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{S}}^{\leq} \psi \iff v(\varphi_0) \wedge \cdots \wedge v(\varphi_{n-1}) \leq v(\psi) \quad \textit{for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{S}) \tag{14}$$

$$\vdash_{\boldsymbol{S}}^{\leq} \psi \iff v(\psi) = 1 \;\; \textit{for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{S}) \tag{15}$$

and

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{S}}^{1} \psi \iff v(\psi) = 1 \textit{ whenever } v(\varphi_0) = \cdots = v(\varphi_{n-1}) = 1\,, \quad \textit{for all } v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{S}) \tag{16}$$

Some elementary properties, independent of the particular $\boldsymbol{S}$, can be immediately derived from the definitions; it is illustrative to make explicit the Hilbert-style and Gentzen-style rules common to all these logics which will be used later on. They are formulated with sentential variables, since they are understood as *rule schemes*, so that satisfying one of them means satisfying all its substitution instances (this proviso is not necessary for Hilbert-style rules, but it makes a difference for the Gentzen-style ones).

Proposition 10 *For each $\boldsymbol{S}$, the logic $Ł_{\boldsymbol{S}}^{\leq}$ satisfies the following rules:*

1. $\dfrac{p_0, \ldots p_{n-1} \vdash q}{p_0^k, \ldots p_{n-1}^k \vdash q^k}$ *for all* $k \geq 1$.
2. $\dfrac{p \dashv\vdash q}{p^k \dashv\vdash q^k}$ *for all* $k \geq 1$.
3. $\dfrac{\vdash p}{\vdash p^k}$ *for all* $k \geq 1$.
4. $\vdash p \to p$.
5. $p \dashv\vdash p \leftrightarrow \top$.
6. $\vdash p \to (q \to p * q)$.
7. $p^s \vdash p^t$ *for all* s, t *with* $s \geq t \geq 1$.

Proof: 1: Let $a_0, \ldots, a_{n-1}, b \in [0,1]$ be such that $a_0 \wedge \ldots \wedge a_{n-1} \leq b$. Since the operation $*$ is monotonic and continuous, for each $k \geq 1$, $a_0^k \wedge \ldots \wedge a_{n-1}^k = (a_0 \wedge \ldots \wedge a_{n-1})^k \leq b^k$. Using this it is straightforward to show 1. 2 is a consequence of a particular case of 1. 3 holds because $1^k = 1$ for all $k \geq 1$, 4 because $a \leftrightarrow a = 1$ for all $a \in [0,1]$, 5 because $a \leftrightarrow 1 = a$ for all $a \in [0,1]$, and 6 because $a \to (b \to a * b) = 1$ for all $a, b \in [0,1]$. Finally, 7 holds because for all $a \in [0,1]$, if $s \geq t \geq 1$ then $a^s \leq a^t$. ∎

In the present context it makes sense to call ***non-trivial*** the subalgebras with more than two elements, that is, with at least one element different from 0 and from 1. The excluded case corresponds to $S = S_2 = \{0, 1\}$, the Boolean algebra associated with classical propositional logic, and indeed $Ł_{\boldsymbol{S}_2} = Ł^{\leq}_{\boldsymbol{S}_2} = \mathcal{CPL}$. Then:

Theorem 11 *For each non-trivial $\boldsymbol{S}$, the logic $Ł_{\boldsymbol{S}}$ is a proper strengthening of the logic $Ł^{\leq}_{\boldsymbol{S}}$ and these two logics have the same theorems. In other words, $Ł_{\boldsymbol{S}}$ is a proper, purely inferential strengthening of $Ł^{\leq}_{\boldsymbol{S}}$. Moreover, for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in Fm$,*

$$\varphi_0, \dots, \varphi_{n-1} \vdash^{\leq}_{\boldsymbol{S}} \psi \iff \vdash^{\leq}_{\boldsymbol{S}} \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \to \psi \tag{17}$$

$$\iff \vdash^{1}_{\boldsymbol{S}} \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \to \psi. \tag{18}$$

Proof: That $Ł_{\boldsymbol{S}}$ is a strengthening of $Ł^{\leq}_{\boldsymbol{S}}$ with the same theorems is a consequence of the general theory of Section 3, or it directly follows from Definition 9. To see that it is a proper one, one can show for instance that

$$p \vdash p * p \tag{19}$$

is a rule of $Ł_{\boldsymbol{S}}$ that is not a rule of $Ł^{\leq}_{\boldsymbol{S}}$: Since $1 * 1 = 1$ in $[\mathbf{0},\mathbf{1}]$, it is clear that (19) is a rule of $Ł_{\boldsymbol{S}}$. But $a \leq a * a$ is only true when $a = 0, 1$, while if $a \neq 0, 1$ then $a > a * a = \max\{0, 2a - 1\}$. Since by assumption $\boldsymbol{S}$ is non-trivial, there are such a in S, therefore (19) is not a rule of $Ł^{\leq}_{\boldsymbol{S}}$. Finally, the last part of the theorem follows directly from Definition 9 and the fact that on any MV-algebra, $a \leq b \iff a \to b = 1$. ∎

The equivalence (17) may be regarded as a kind of ***Weak Deduction Theorem***. Together with (18), these equivalences seem to suggest there is no particular interest in $Ł^{\leq}_{\boldsymbol{S}}$, as it can be reduced to $Ł_{\boldsymbol{S}}$. However, the ordering structure of the real line is so natural that often it is simpler to work with $Ł^{\leq}_{\boldsymbol{S}}$ than with $Ł_{\boldsymbol{S}}$. Only the traditionally more accepted scheme of preserving truth has come to make $Ł_{\boldsymbol{S}}$ appear as a more natural logic than $Ł^{\leq}_{\boldsymbol{S}}$. To a certain extent, I would agree with the reverse judgement, and moreover in Theorem 15 below I show that for a finite $\boldsymbol{S}$ the logic $Ł_{\boldsymbol{S}}$ can be reduced to $Ł^{\leq}_{\boldsymbol{S}}$ in a similar way.

The general theory summarized in Section 3 already explains the main algebraic properties of the logics. By their very definition the logics $Ł^{\leq}_{\boldsymbol{S}}$ are semilattice-based with respect to the single algebra $\boldsymbol{S}$, and hence with respect to the variety it generates; then Theorems 2, 3 and 4 automatically yield:

Proposition 12 *For each subalgebra $\boldsymbol{S}$ of $[\mathbf{0},\mathbf{1}]$, the logic $Ł^{\leq}_{\boldsymbol{S}}$ is selfextensional and has conjunction. Its algebraic counterpart is $\mathsf{Alg}Ł^{\leq}_{\boldsymbol{S}} = \mathsf{V}(\boldsymbol{S})$, and on each algebra of this class the $Ł^{\leq}_{\boldsymbol{S}}$-filters coincide with the lattice filters. A generalized matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full model of $\mathcal{S}$ if and only if there is a strict surjective homomorphism from $\langle \boldsymbol{A}, \mathcal{C} \rangle$ onto a generalized matrix of the form $\langle \boldsymbol{B}, \mathcal{F}ilt(\boldsymbol{B}) \rangle$ with $\boldsymbol{B} \in \mathsf{V}(\boldsymbol{S})$.* ∎

Recall from Section 3 that $\mathcal{F}ilt(\boldsymbol{B})$ is the set of all lattice-filters of $\boldsymbol{B}$. Hence in case $\boldsymbol{B} \in \mathsf{V}(\boldsymbol{S})$ then $\mathcal{F}ilt(\boldsymbol{B}) = \mathcal{F}i_{Ł^{\leq}_{\boldsymbol{S}}} \boldsymbol{B}$.

Proposition 13 *For each non-trivial $\boldsymbol{S}$, the logic $Ł_{\boldsymbol{S}}$ has conjunction, and is not selfextensional.*

Proof: Since having conjunction is expressed by Hilbert-style rules, it is a property inherited by strengthenings of any kind, so $Ł_{\boldsymbol{S}}$ has it because $Ł_{\boldsymbol{S}}^{\leq}$ has it, by Proposition 12. Now to show that $Ł_{\boldsymbol{S}}$ is not selfextensional, let p be any variable, and consider the formulas $\varphi = p$ and $\psi = p * p$. Since for all $a \in [0,1]$, $a = 1 \iff a * a = 1$, $\varphi \dashv\vdash_{\boldsymbol{S}}^{1} \psi$ for any $\boldsymbol{S}$. Now let $a \in S$ be such that $0 < a \leq 1/2$ (it exists because $\boldsymbol{S}$ is nontrivial and negation makes it symmetric with respect to $1/2$). Then $\neg a \neq 1$, $a * a = 0$ and $\neg(a * a) = 1$. This implies that $\neg\psi \not\dashv\vdash_{\boldsymbol{S}}^{1} \neg\varphi$, hence the interderivability relation of $Ł_{\boldsymbol{S}}$ is not a congruence with respect to negation, therefore this logic is not selfextensional. ∎

An alternative way of proving that the $Ł_{\boldsymbol{S}}$ are not selfextensional would be to use Proposition 5 and Theorem 11; but this would only work in case $Ł_{\boldsymbol{S}}^{\leq}$ is protoalgebraic, something we do not know by now (and, as is shown in the next section, is not always the case), so I had to give a direct, general argument.

Proposition 12 determines the algebraic counterparts of all $Ł_{\boldsymbol{S}}^{\leq}$ in both the traditional and the more abstract senses. Another typical task of Abstract Algebraic Logic is to *classify* the logics according to several criteria, notably with respect the so-called hierarchy outlined in Section 1. However, to go further in this direction it is useful (or perhaps indispensable) to treat separately the cases where $\boldsymbol{S}$ is finite, which have a perfectly standard behaviour, from the cases where $\boldsymbol{S}$ is infinite.

Therefore *I am going to assume in the rest of this section* that for some $m \geq 2$, $\boldsymbol{S} = \boldsymbol{S}_m$, the subalgebra of $[\mathbf{0},\mathbf{1}]$ with m elements, that is, with universe $S_m = \{0, \frac{1}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}$; here it is more practical to write $Ł_m^{\leq} = \langle \boldsymbol{Fm}, \vdash_m^{\leq} \rangle$ and $Ł_m = \langle \boldsymbol{Fm}, \vdash_m \rangle$ instead of $Ł_{\boldsymbol{S}_m}^{\leq}$ and $Ł_{\boldsymbol{S}_m}$, respectively. WÓJCICKI [43], see also [44, Theorem 4.3.3], showed that the logic here denoted by $Ł_m$ fully coincides with the one axiomatized by the tautologies of what is usually called ***the*** m***-valued Łukasiewicz logic*** plus the rule of Modus Ponens. In [44, Theorem 4.3.8] it is shown that $Ł_m$ is an "implicative logic" in the sense of [34], therefore by the observation in page 41 of [3] it is a strongly, regularly and finitely algebraizable logic, and by the same result in [44] and Corollary 5.3 of [3] it follows that its equivalent algebraic semantics is the variety $\mathsf{Alg}Ł_m = \mathsf{V}(\boldsymbol{S}_m)$, often called ***the variety of*** m***-valued MV-algebras***, see [8, Definition 8.5.2]. For $m > 2$ this variety has been axiomatized in different ways, see [8, Theorem 8.5.1]. In the case $m = 2$ one gets $S_2 = \{0,1\}$, the two-element Boolean algebra, $Ł_2 = Ł_2^{\leq} = \mathcal{CPL}$, and $\mathsf{Alg}Ł_2 = \mathsf{V}(\boldsymbol{S}_2)$ is the variety of all Boolean algebras.

To see that the pairs $\big(Ł_m^{\leq}, Ł_m\big)$ fit into the general framework of the preceding sections, one has to obtain directly a few properties of the less-known logics $Ł_m^{\leq}$. That they are semilattice-based with respect to $\boldsymbol{S}_m$ was already observed in [44, Section 4.3.14] and in [33]. As logics preserving degrees of truth they were briefly studied in GIL's unpublished Ph. D. Thesis [21], in the context of many-sided sequent calculi; one of his results adds to the general properties of Theorem 11 and Proposition 12, allowing to be more precise about their classification in the finite case:

Theorem 14 (GIL) *For each* $m \geq 2$ *the logic* $Ł_m^{\leq}$ *is finitely equivalential, the formula* $(p \leftrightarrow q)^{m-1}$ *being its single equivalence formula.*

Proof: The simplest way to show this is to check that the proposed equivalence formula satisfies the six syntactical conditions from [10, Definition I.10]; see also [11, Chapter 3]:

(E1) $\vdash_m^{\leq} (p \leftrightarrow p)^{m-1}$

(E2) $(p \leftrightarrow q)^{m-1} \vdash_m^{\leq} (q \leftrightarrow p)^{m-1}$

(E3) $(p \leftrightarrow q)^{m-1}, (q \leftrightarrow r)^{m-1} \vdash_m^{\leq} (p \leftrightarrow r)^{m-1}$

(E4) $(p \leftrightarrow q)^{m-1} \vdash_m^{\leq} (\neg p \leftrightarrow \neg q)^{m-1}$

(E5) $(p \leftrightarrow q)^{m-1}, (p' \leftrightarrow q')^{m-1} \vdash_m^{\leq} \big((p \to p') \leftrightarrow (q \to q')\big)^{m-1}$

(E6) $p, (p \leftrightarrow q)^{m-1} \vdash_m^{\leq} q$

(E1) is a consequence of 10.4 and 10.2. (E2) by the symmetry in the truth-valued function of $\leftrightarrow$. (E3) is proved by taking into account that if $a \in S_m$ and $a \neq 1$ then $a^{m-1} = 0$ (actually, $a^n = 0$ for all $n \geq m-1$) while $1^m = 1$, and that $a \to b = 1$ if and only if $a \leq b$; hence $a \leftrightarrow b = 1$ if and only if $a = b$. Then take any $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{S}_m)$. If $v(p) \neq v(q)$ or $v(q) \neq v(r)$ then $v\big((p \leftrightarrow q)^{m-1}\big) \wedge v\big((q \leftrightarrow r)^{m-1}\big) = 0 \leq v\big((p \leftrightarrow r)^{m-1}\big)$. If $v(p) = v(q) = v(r)$ then $v\big((p \leftrightarrow q)^{m-1}\big) \wedge v\big((q \leftrightarrow r)^{m-1}\big) = 1 = v\big((p \leftrightarrow r)^{m-1}\big)$. This shows that (E3) holds. Similar reasonings prove (E4) and (E5). Finally, to show (E6), if $v(p) \neq v(q)$ then $v(p) \wedge v\big((p \leftrightarrow q)^{m-1}\big) = 0 \leq v(q)$, while if $v(p) = v(q)$ then $v(p) \wedge v\big((p \leftrightarrow q)^{m-1}\big) = v(p) \wedge 1 = v(p) = v(q)$. ∎

Every equivalential logic is a fortiori protoalgebraic [11, page 185], hence Proposition 5 and Theorems 6 and 7 apply. As one of the applications of these results one obtains:

Theorem 15 *For every $m \geq 2$, the logic $Ł_m$ is the "strong version" of $Ł_m^{\leq}$ in the sense of Section 3, that is, $Ł_m = \big(Ł_m^{\leq}\big)^+$, and for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in Fm$,*

$$\varphi_0, \dots, \varphi_{n-1} \vdash_m \psi \iff \varphi_0^{m-1}, \dots, \varphi_{n-1}^{m-1} \vdash_m^{\leq} \psi.$$

Proof: The first part follows from Theorem 7, because by the preceding result the logic $Ł_m^{\leq}$ is equivalential, and it has been proved in Theorem 3.8 of [23] that for every $m \geq 2$, $\mathbf{Q}(\boldsymbol{S}_m) = \mathbf{V}(\boldsymbol{S}_m)$; hence the equivalent condition in Theorem 7 holds in this case, and therefore the strong version of $Ł_m^{\leq}$ coincides with $Ł_m$. The second part follows from the first one plus Theorem 8, after taking into account that the set $X(p)$ mentioned in this last result has the form $X(p) = \Delta(p, \top)$ for any set Δ of equivalence formulas for the weak logic, here $Ł_m^{\leq}$. Hence here $X(p) = \big\{(p \leftrightarrow \top)^{m-1}\big\}$. However, expression (13) clearly shows that any $Ł_m^{\leq}$-equivalent set can replace $X(p)$. By 10.5 and 10.2, $(p \leftrightarrow \top)^{m-1} \dashv\vdash_m^{\leq} p^{m-1}$, therefore one can equally use $X(p) = \{p^{m-1}\}$. ∎

Corollary 16 *For every $m > 2$, the logic $Ł_m^{\leq}$ is not weakly algebraizable, hence a fortiori it is not algebraizable in any sense, while $Ł_2^{\leq} = Ł_2 = \mathcal{CPL}$ (classical propositional logic) is.*

Proof: As was said in Section 3, weakly algebraizable logics coincide with their strong version. Hence, by Theorem 15, if $Ł_m^{\leq}$ is so, then $Ł_m^{\leq} = (Ł_m^{\leq})^+ = Ł_m$. But by Theorem 11 $Ł_m$ is a proper strengthening of $Ł_m^{\leq}$, hence different from it. Therefore $Ł_m^{\leq}$ cannot be weakly algebraizable. All this concerns the case $m > 2$, while for $m = 2$ the properties of $\mathcal{CPL}$ are well-known. ∎

This completes the classification of the logics $Ł_m^{\leq}$ in the hierarchy. Moreover, $Ł_m$ *is the "strong version" of* $Ł_m^{\leq}$ *in two very different senses*: On one side, $Ł_m$ is the logic defined by preserving truth from the same structure (the m-valued algebra $\boldsymbol{S}_m$) with respect to which $Ł_m^{\leq}$ is defined by preserving degrees of truth. On the other side, the logic $Ł_m$ is the strong version, in the general sense of Abstract Algebraic Logic, of the logic $Ł_m^{\leq}$, that is, $Ł_m$ is determined by the Leibniz filters of $Ł_m^{\leq}$. I now show that there is more: Leibniz filters of $Ł_m^{\leq}$ not only constitute a defining matrix semantics for $Ł_m$ but they are exactly *all* its filters; moreover, they can be nicely characterised on arbitrary algebras of the signature of MV-algebras (i.e., not only on an MV-algebra):

Theorem 17 *Let F be any $Ł_m^{\leq}$-filter on any algebra $\boldsymbol{A}$ of the signature of MV-algebras. Then the following conditions are equivalent:*

(i) *F is a Leibniz filter of $Ł_m^{\leq}$.*

(ii) *F is an $Ł_m$-filter.*

(iii) *For all $a, b \in A$, if $a \in F$ and $a \to b \in F$ then $b \in F$.*

(iv) *For all $a, b \in A$, if $a \in F$ and $b \in F$ then $a * b \in F$.*

(v) *For all $a \in A$, if $a \in F$ then $a * a \in F$.*

(vi) *For all $a \in A$, if $a \in F$ then $a^{m-1} \in F$.*

Proof: (i)⇒(ii) by Theorem 15 and the general definition of $\left(Ł_m^{\leq}\right)^{+}$ as the logic determined by the Leibniz filters of $Ł_m^{\leq}$. (ii)⇒(iii) as it is easy to check that $p, p \to q \vdash q$ is a rule of $Ł_m$. (iii)⇒(iv) because the formula $p \to (q \to p * q)$ is a theorem of $Ł_m^{\leq}$, by 10.6, and F is a filter of this logic. (v) is a particular case of (iv). From (v) one gets, by iteration, that if $a \in F$ then $a^{2^k} \in F$ for all $k \neq 0$, in particular for $2^k \geq m-1$; but then by 10.7 the rule $p^{2^k} \vdash p^{m-1}$ holds for $Ł_m^{\leq}$, so $a^{m-1} \in F$ because F is by assumption an $Ł_m^{\leq}$-filter. Finally only the proof of (vi)⇒(i) remains. By Theorem 14 we can use the characterization (2) of the Leibniz congruence of any $Ł_m^{\leq}$-filter F on an arbitrary algebra $\boldsymbol{A}$; this means that for all $a, b \in A$, $\langle a, b\rangle \in \boldsymbol{\Omega_A}(F) \iff (a \leftrightarrow b)^{m-1} \in F$ for any $Ł_m^{\leq}$-filter F. So assume now that F is an $Ł_m^{\leq}$-filter on $\boldsymbol{A}$ satisfying condition (vi), and let G be any $Ł_m^{\leq}$-filter on the same $\boldsymbol{A}$ such that $\boldsymbol{\Omega_A}(G) = \boldsymbol{\Omega_A}(F)$. Take $a \in F$; by assumption $a^{m-1} \in F$. But as observed during the proof of Theorem 15, $(p \leftrightarrow \top)^{m-1} \dashv\vdash_m^{\leq} p^{m-1}$, and F is closed under all rules of $Ł_m^{\leq}$, therefore $(a \leftrightarrow 1)^{m-1} \in F$, that is, $\langle a, 1\rangle \in \boldsymbol{\Omega_A}(F) = \boldsymbol{\Omega_A}(G)$. Since $1 \in G$, by compatibility also $a \in G$. This shows that $F \subseteq G$ and completes the proof that F is Leibniz. ∎

Corollary 18 *The logic $Ł_m$ is the inferential strengthening of the logic $Ł_m^{\leq}$ by any of the following proper rules:*

$$p\,, p \to q \vdash q \qquad \text{(i.e., the rule of Modus Ponens)}$$
$$p\,, q \vdash p * q$$
$$p \vdash p * p$$
$$p \vdash p^{m-1}$$

∎

One sometimes thinks of $Ł_m^{\leq}$ as "$Ł_m$ minus Modus Ponens". It is an open problem to find a Hilbert-style presentation of $Ł_m^{\leq}$; if one is found then adding to it any of the rules in Corollary 18 will yield one of $Ł_m$. Unfortunately, the existing presentations of $Ł_m$ do not help in this problem because they have only one rule of inference, namely Modus Ponens. A Gentzen-style presentation of $Ł_m^{\leq}$ has been proposed, without proof, in [22]. There are other syntactical relations between the two logics; besides that of Theorem 15 there is the following particular case of Theorem 11: For all $\varphi_0, \dots, \varphi_{n-1}, \psi \in Fm$,

$$\varphi_0, \dots, \varphi_{n-1} \vdash_m^{\leq} \psi \iff \vdash_m \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \to \psi. \tag{20}$$

A consequence of these relations is:

Theorem 19 *The sets of logics* $\{Ł_m^{\leq} : m \geq 2\}$ *and* $\{Ł_m : m \geq 2\}$ *are isomorphic ordered sets when ordered under the "strengthening" relation, and have* $\mathcal{CPL}$ *(classical logic) as their common upper bound. These sets are lattices where the infimum and supremum operations are given by the following arithmetical operations on subindexes:*

$$m \wedge k = \operatorname{lcm}(m-1, k-1) + 1 \tag{21}$$

$$m \vee k = \gcd(m-1, k-1) + 1 \tag{22}$$

Proof: The "strengthening" relation is clearly an ordering relation between logics, that is, between the consequence relations considered set-theoretically. From (20) it follows that for all $m, k \geq 2$, $\vdash_m^{\leq} \subseteq \vdash_k^{\leq}$ if and only if the set of theorems of $\vdash_m$ is included in the set of theorems of $\vdash_k$. It is well-known [24, Theorem 9.1.2] that this happens if and only if $k-1$ divides $m-1$, which is equivalent to saying that $\boldsymbol{S}_k$ is a subalgebra of $\boldsymbol{S}_m$. By the way the logics are defined from the algebras, this obviously implies that $\vdash_m \subseteq \vdash_k$. To show that this in turn implies that $\vdash_m^{\leq} \subseteq \vdash_k^{\leq}$ one needs just to apply the property (20), which reduces derivability in $Ł_m^{\leq}$ to theoremhood in $Ł_m$, and the same for k. This completes the proof that the two ordered sets are isomorphic. Using the same fundamental properties it is easy to show that both sets are lattices with the lattice operations corresponding to the operations on indexes expressed by (21) and (22), that is, $Ł_s^{\leq} = Ł_m^{\leq} \wedge Ł_k^{\leq}$ if and only if $Ł_s = Ł_m \wedge Ł_k$ if and only if $s = \operatorname{lcm}(m-1, k-1)+1$, and $Ł_s^{\leq} = Ł_m^{\leq} \vee Ł_k^{\leq}$ if and only if $Ł_s = Ł_m \vee Ł_k$ if and only if $s = \gcd(m-1, k-1) + 1$. Finally, clearly $Ł_2^{\leq} = Ł_2 = \mathcal{CPL}$, and since 1 divides $m-1$ for all $m \geq 2$, it is clear that $\mathcal{CPL}$ is the common upper bound of both sets, in fact their maximum. ∎

In the next section the lower bounds of each of the two sets will be determined.

5 Logics defined from infinite subalgebras of the real unit interval

Logics of this kind have been analysed in [15, Section 3C] as examples of the notion of "strong version" of a protoalgebraic logic, in a context where the logics need not necessarily be finitary, that is, they are what in Section 1 was called "consequence relations"; they were also denoted by $Ł_{\boldsymbol{S}}^{\leq}$ and $Ł_{\boldsymbol{S}}$, as in the present paper, but the reader should not be confused: the "logics" of [15] are slightly different from those of the present paper. More precisely:

Theorem 20 *Let $\boldsymbol{S}$ be a non-trivial subalgebra of* $[\mathbf{0},\mathbf{1}]$ *and consider the two consequence relations defined by using the non-finitized schemes* (3) *and* (5) *by taking* $\boldsymbol{A} = \boldsymbol{S}$ *and* $Val = \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{S})$. *Then each of these consequence relations is finitary if and only if the algebra* $\boldsymbol{S}$ *is finite.*

Proof: I have already observed in Section 2 that these consequences are those defined by a single matrix (6) and by a set of matrices (7), respectively. Now, if $\boldsymbol{S}$ is finite then the matrix (6) is finite, while the set of matrices (7) is a finite set of finite matrices. So both consequence relations are defined by a finite set of finite matrices, and it is well-known (see [44, Theorem 4.1.7] for instance) that such a consequence is finitary. Now assume that the algebra $\boldsymbol{S}$ is infinite. By McNaughton's Theorem [24, Theorem 9.1.8], it is easy to see that for each $k \in \omega$ there exists a formula $\varphi_k(p)$ in one variable p such that for all $a \in [0,1]$, $\varphi_k^{[\mathbf{0},\mathbf{1}]}(a) = 0$ iff $a \leq \frac{k+1}{k+3}$ and $\varphi_k^{[\mathbf{0},\mathbf{1}]}(a) = 1$ iff $a \geq \frac{k+2}{k+3}$. Now take $\Sigma = \{\varphi_k(p) : k \in \omega\}$ and let $\boldsymbol{S} \subseteq [\mathbf{0},\mathbf{1}]$ be an infinite subalgebra. The consequence relations defined from $\boldsymbol{S}$ by (3) and (5) will be represented inside this proof by $\vdash^1$ and $\vdash^{\leq}$ respectively, as in Section 2. Clearly, $\vdash^{\leq} \subseteq \vdash^1$. It is straightforward to check that $\Sigma \vdash^1 p$ and $\Sigma \vdash^{\leq} p$. However, if $\Sigma_0 \subseteq \Sigma$ is finite, then $\Sigma_0 \nvdash^1 p$: This follows from the fact, proved in Proposition 3.5.3 of [8], that S, being infinite, must be a dense subset of $[0,1]$; therefore for each $n \in \omega$ there is some $a_n \in S$ with $\frac{n+2}{n+3} < a_n < 1$, hence by the construction of the formulas $\varphi_k^{\boldsymbol{S}}(a_n) = \varphi_k^{[\mathbf{0},\mathbf{1}]}(a_n) = 1$ for all $k \leq n$ while $a_n \neq 1$. This shows that $\Sigma_0 \nvdash^1 p$ and a fortiori $\Sigma_0 \nvdash^{\leq} p$. Thus, neither of the two consequence relations is finitary. ∎

By Definition 9, the logics $\boldsymbol{Ł}_{\boldsymbol{S}}^{\leq}$ and $\boldsymbol{Ł}_{\boldsymbol{S}}$ in the present paper are finitary, hence they do not coincide with the consequence relations denoted by the same symbols in [15]. Moreover, since the general framework of semilattice-based logics summarized in Section 3 relies heavily on the logics being finitary, this also says that whatever results one can obtain from it cannot be based on those of [15, Section 3C].

The logics defined from the whole algebra $[\mathbf{0},\mathbf{1}]$ will play a special role from now on, so for ease of notation I will denote them by $Ł_\infty^{\leq}$ and $Ł_\infty$ instead of $Ł_{[\mathbf{0},\mathbf{1}]}^{\leq}$ and $Ł_{[\mathbf{0},\mathbf{1}]}$ respectively. Notice that $Ł_\infty$ is the *finitary* logic defined by the matrix $\big\langle [\mathbf{0},\mathbf{1}] , \{1\} \big\rangle$; that the consequence relation defined from this matrix is non-finitary (as Theorem 20 states for all infinite $\boldsymbol{S}$) was first noticed by Wójcicki in [43]. The tautologies of this matrix form what is usually called ***the infinite-valued Łukasiewicz logic***. After their axiomatization by Rose and Rosser, and independently by Chang, it was proved by Hay [26] that $Ł_\infty$ coincides with the logic axiomatized by those tautologies and the rule of Modus Ponens; see also [8, p. 101] and [25, Theorem 3.2.13]. Rodríguez, Torrens and Verdú proved in [37] that it is finitely, strongly and regularly algebraizable, that its equivalent algebraic semantics is the class MV of all MV-algebras and that on each $\boldsymbol{A} \in \mathsf{MV}$ the $Ł_\infty$-filters coincide with the ***implicative filters***, i.e., those $F \subseteq A$ such that $1 \in F$ and F is closed under Modus Ponens (if $a, a \to b \in F$ then b$\in F$). As a consequence, $Ł_\infty$ is the logic defined by the class of matrices

$$\big\{ \langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{MV} , F \subseteq A \text{ is an implicative filter} \big\}. \tag{23}$$

The proofs in [37] show that the set of equivalence formulas for $Ł_\infty$ is the set $\{p \to q , q \to p\}$, which can be replaced in this function by the single formula $p \leftrightarrow q$.

Recall that, as a particular case of Theorem 11, $Ł_\infty^{\leq}$ and $Ł_\infty$ are linked by the relation

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_\infty^{\leq} \psi \iff \vdash_\infty \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \rightarrow \psi. \tag{24}$$

Concerning the other logics defined by preservation of truth from infinite subalgebras of $[\mathbf{0},\mathbf{1}]$, one finds:

Theorem 21 *The logics $Ł_{\boldsymbol{S}}$ for an infinite $\boldsymbol{S} \subseteq [\mathbf{0},\mathbf{1}]$ depend only on the rationals contained in $\boldsymbol{S}$. That is, if $\boldsymbol{S}$ and $\boldsymbol{T}$ are two infinite subalgebras of $[\mathbf{0},\mathbf{1}]$ then $Ł_{\boldsymbol{S}} = Ł_{\boldsymbol{T}}$ if and only if $S \cap \mathbb{Q} = T \cap \mathbb{Q}$. Moreover, all the logics $Ł_{\boldsymbol{S}}$ have the same theorems for all infinite subalgebras $\boldsymbol{S}$ of $[\mathbf{0},\mathbf{1}]$.*

Proof: It is obvious that if $\boldsymbol{S} \subseteq [\mathbf{0},\mathbf{1}]$ then the logic $Ł_{\boldsymbol{S}}$ is a strengthening of $Ł_\infty$. Hence all these $Ł_{\boldsymbol{S}}$ are algebraizable with equivalent algebraic semantics the quasivariety $\mathsf{Q}(\boldsymbol{S})$, which is a subclass of the variety MV, and with the same equivalence formula $p \leftrightarrow q$. Now, in Theorem 2.9 of [23] it is shown that $\mathsf{Q}(\boldsymbol{S}) = \mathsf{Q}(\boldsymbol{T})$ if and only if $S \cap \mathbb{Q} = T \cap \mathbb{Q}$. Since an algebraizable logic is completely determined by its equivalent algebraic semantics and its defining equations, this establishes the first part of the statement. Now, again by algebraizability, the theorems of $Ł_{\boldsymbol{S}}$ are translated into the equations holding in the quasivariety $\mathsf{Q}(\boldsymbol{S})$, which are those holding in the variety generated by $\boldsymbol{S}$. But it is a straightforward consequence of a well-known theorem of LINDENBAUM [29, Theorem 16] that this variety is always (for an infinite $\boldsymbol{S}$) the whole variety MV, hence all these $Ł_{\boldsymbol{S}}$ have the same theorems. This establishes the second part of the theorem. ∎

Thus, only the logics defined by preservation of truth inside a *finite* truth-value-algebra have particular theorems. This constitutes another proof that the topic of the present section can only be of interest for those interested in the *inferential* side of logic: for infinite $\boldsymbol{S}$, the logics $Ł_{\boldsymbol{S}}$ are purely inferential strengthenings of $Ł_\infty$. Actually, the general theory of algebraizability shows that any set of quasiequations defining $\mathsf{Q}(\boldsymbol{S})$ relatively to MV yields, when translated through the equivalence formula, an axiomatization of the additional rules of $Ł_{\boldsymbol{S}}$ with respect to $Ł_\infty$. In particular, in the cases where $\boldsymbol{S}$ contains all rationals in $[0,1]$ the $Ł_{\boldsymbol{S}}$ is equal to $Ł_\infty$.

For logics defined by preservation of truth, there is not a one-to-one correspondence between logics and infinite subalgebras of $[\mathbf{0},\mathbf{1}]$. The case of the logics defined by preservation of degrees of truth the situation is even worse:

Theorem 22 *If $\boldsymbol{S}$ is an infinite subalgebra of $[\mathbf{0},\mathbf{1}]$ then $Ł_{\boldsymbol{S}}^{\leq} = Ł_\infty^{\leq}$.*

Proof: This is due to the fact that by definition all these logics are semilattice-based with respect to $\boldsymbol{S}$ and hence, by the properties summarized in Section 3, with respect to $\mathsf{V}(\boldsymbol{S})$. As I have already recalled in the previous proof, by LINDENBAUM's result [29, Theorem 16], for an infinite $\boldsymbol{S}$ all these varieties are equal and coincide with MV. Therefore all these logics coincide with the logic that is semilattice-based with respect to MV, that is, with $Ł_\infty^{\leq}$. ∎

Thus *there is only one logic defined by preservation of an infinite number of degrees of truth in* $[\mathbf{0},\mathbf{1}]$, namely $Ł_\infty^{\leq}$. Some of its properties follow from the general theory of Section 3: it is selfextensional and has conjunction, and $\mathsf{Alg}Ł_\infty^{\leq} = \mathsf{MV}$; moreover, on each of these algebras its filters are the lattice filters. As a consequence, in parallel with (23), $Ł_\infty^\leq$ is the logic defined by the class of matrices

$$\big\{\langle \boldsymbol{A}\,, F\rangle : \boldsymbol{A} \in \mathsf{MV}\,, F \subseteq A \text{ is a lattice filter}\big\}. \tag{25}$$

From this one can also infer that $Ł_\infty^\leq$ it coincides with the logic denoted as C_V in [22]. In this paper a Gentzen calculus is presented that is claimed (without proof) to be a Gentzen-style presentation of this logic; a reasonable conjecture is that the Gentzen system defined by this calculus is the (unique) Gentzen system fully adequate for $Ł_\infty^\leq$, which exists by the general theory of [16].

By Theorem 11, each logic $Ł_{\boldsymbol{S}}$ defined by the truth-preserving scheme is different from the corresponding logic $Ł_{\boldsymbol{S}}^\leq$ defined by the preservation of degrees of truth scheme. Corollary 18 provides an exact, Hilbert-style relationship between each two them in the case $\boldsymbol{S}$ is finite, and in Theorem 19 a global relationship between the two families has been established. By Theorems 21 and 22, there is no hope that such results can be replicated in the infinite case, for most of the logics of the first kind coincide, as do *all* those of the second. It is only reasonable to expect that something similar holds for the weakest cases of $Ł_\infty$ and $Ł_\infty^\leq$. However:

Theorem 23 (GIL, TORRENS, VERDÚ) *The logic $Ł_\infty^\leq$ is not protoalgebraic.*

Proof: This result is in fact a corollary to Theorem 5 of [22] that characterizes the protoalgebraic strengthenings of the logic denoted there as C_V, which as observed before should be equal to the logic here denoted as $Ł_\infty^\leq$. Since [22] appears in a proceedings volume very difficult to find, and moreover contains no proofs, it may be of interest to give here a direct proof.

An element a of an MV-algebra $\boldsymbol{A}$ is *archimedean*, according to [8, Corollary 6.2.4], when there is an integer $n \geq 1$ such that $\neg a \vee n\,a = 1$; an MV-algebra all whose elements are archimedean is called *hyperarchimedean* [8, Definition 6.3]. Let $\boldsymbol{A}$ be any such algebra, for instance the direct product of the algebras $\boldsymbol{S}_m$ for all $m \geq 2$ (see [8, Chapter 6] or [40, Section 3.A]) and let $a \in A$ be any non-archimedean element of $\boldsymbol{A}$. Let $F = \mathrm{Fi}(\neg a)$ be the implicative filter generated by $\neg a$; this set is both an $Ł_\infty^\leq$-filter (thus, a lattice filter) and an $Ł_\infty$-filter. This last logic is algebraizable, hence equivalential, therefore (2) can be used with $\Delta(p\,,q) = \{p \to q\,, q \to p\}$. Since $a \to 0 = \neg a \in F$ by construction, and $0 \to a = 1 \in F$ because all filters contain the maximum, by (2) $\langle a, 0\rangle \in \boldsymbol{\Omega_A}(F)$. If by contradiction one assumes that $Ł_\infty^\leq$ is protoalgebraic, then by [11, Corollary 1.1.11] from the fact that $\langle a, 0\rangle \in \boldsymbol{\Omega_A}(F)$ it follows that a and 0 must belong to the same $Ł_\infty^\leq$-filters on $\boldsymbol{A}$ that contain F; in particular this implies that $0 \in \mathcal{F}ilt(F\,,a)$, the lattice filter generated by $F \cup \{a\}$. Since F is itself a lattice filter, by standard lattice-filter generation this means that $0 = b \wedge a$ for some $b \in F = \mathrm{Fi}(\neg a)$. Now the characterization of implicative filters in MV-algebras found in [8, Proposition 4.2.9] or [40, 1.19] can be used, which is an algebraic version of the Local Deduction Theorem for $Ł_\infty$, that is, that $\mathrm{Fi}(\neg a) = \{c \in A : (\neg a)^n \leq c \text{ for some } n \geq 1\}$. Hence $(\neg a)^n \leq b$ for some $n \geq 1$ and thus $(\neg a)^n \wedge a = 0$, which by the De Morgan Laws for both the pairs $(\wedge\,,\vee)$ and $(\oplus\,,*)$ is equivalent to $n\,a \vee \neg a = 1$, which would mean that a is archimedean, against the assumption. Therefore $Ł_\infty^\leq$ cannot be protoalgebraic. ∎

Thus, the general theory developed in [15] for protoalgebraic logics cannot be applied to find out an abstract relationship between $Ł_\infty^\leq$ and $Ł_\infty$. We can instead consider the following, truly general definition of a "strong version" of a logic: Given an arbitrary logic $\mathcal{S}$, one can always consider the logic $\mathcal{S}^{\min}$ defined by the class of matrices

$$\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{Alg}\mathcal{S}\,, F = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\},$$

that is, the logic defined by taking only the smallest matrix on each $\mathcal{S}$-algebra. Since each of these matrices is of course an $\mathcal{S}$-matrix, the logic $\mathcal{S}^{\min}$ is a strengthening of $\mathcal{S}$. And in the present case one obtains:

Theorem 24 $(Ł_\infty^{\leq})^{\min} = Ł_\infty$.

Proof: Since $\mathsf{Alg}Ł_\infty^{\leq} = \mathsf{MV}$ and on any MV-algebra the least $Ł_\infty^{\leq}$-filter is $\{1\}$, the logic $(Ł_\infty^{\leq})^{\min}$ is the logic defined by the class of matrices $\big\{\big\langle \boldsymbol{A}\,, \{1\}\big\rangle : \boldsymbol{A} \in \mathsf{MV}\big\}$. But MV is the equivalent algebraic semantics for $Ł_\infty$ and $\{1\}$ is the designated filter on each $\boldsymbol{A} \in \mathsf{MV}$, therefore $(Ł_\infty^{\leq})^{\min}$ coincides with the logic $Ł_\infty$. ∎

Thus the relation between $Ł_\infty^{\leq}$ and $Ł_\infty$ is parallel to that found between $Ł_m^{\leq}$ and $Ł_m$ for each finite m: On one hand $Ł_\infty$ is the logic defined by preserving truth from the same structure (the real unit interval) with respect to which $Ł_\infty^{\leq}$ is defined by preserving degrees of truth. On the other side, the logic $Ł_\infty$ is the strong version, in a different abstract sense, of the logic $Ł_\infty^{\leq}$. There is also a version of Corollary 15:

Theorem 25 *The logic $Ł_\infty$ is the inferential strengthening of the logic $Ł_\infty^{\leq}$ by any of the following proper rules*

$$p\,, p \to q \vdash q \qquad \text{(i.e., the rule of Modus Ponens)}$$
$$p\,, q \vdash p * q$$
$$p \vdash p * p$$

or by the infinite set of proper rules

$$p \vdash p^n \quad \text{for all } n \geq 2\,.$$

Proof: $\mathsf{Alg}Ł_\infty^{\leq} = \mathsf{Alg}Ł_\infty = \mathsf{MV}$ and by Theorem 2.22 of [14], each logic $\mathcal{S}$ is complete with respect to the class of matrices $\big\{\langle \boldsymbol{A}\,, F\rangle : \boldsymbol{A} \in \mathsf{Alg}\mathcal{S}\;,\; F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$; hence it is enough to compare the filters of the two logics in the class of MV-algebras. It is true that an $Ł_\infty^{\leq}$-filter on an MV-algebra is just a lattice filter, and that an $Ł_\infty$-filter on an MV-algebra is just an implicative filter. And as a consequence of Proposition 9 of [18], see also [8, Lemma 4.2.7], a lattice filter on an MV-algebra is an implicative filter if and only if it is closed under Modus Ponens, and also if and only if it is closed under the rule $p, q \vdash p * q$. This shows that the first two rules do axiomatize $Ł_\infty$ relatively to $Ł_\infty^{\leq}$. In order to show that the apparently weaker rule $p \vdash p * p$ also defines $Ł_\infty$ from $Ł_\infty^{\leq}$, I show that an $Ł_\infty^{\leq}$-theory that is closed under it is also closed under the rule $p, q \vdash p * q$: Let $\mathit{\Gamma}$ be any such theory, and $\varphi, \psi \in \mathit{\Gamma}$. Take any $t \in [0, 1]$ and any $v \in \mathrm{Hom}(\boldsymbol{Fm}, [\mathbf{0{,}1}])$ such that $v(\gamma) \geq t$ for all $\gamma \in \mathit{\Gamma}$. In particular $v(\varphi)\,, v(\psi) \geq t$. Since $[0, 1]$ is linearly ordered, let's say $v(\varphi) \leq v(\psi)$. By monotonicity of

$*$, $v(\varphi * \varphi) = v(\varphi) * v(\varphi) \leq v(\varphi) * v(\psi) = v(\varphi * \psi)$. But by assumption $\varphi \in \Gamma$ implies $\varphi * \varphi \in \Gamma$, and one concludes that $v(\varphi * \psi) \geq t$. Since t and v are arbitrary, this shows that $\Gamma \vdash^{\leq}_{\infty} \varphi * \psi$, which implies $\varphi * \psi \in \Gamma$ because Γ is assumed to be an $Ł^{\leq}_{\infty}$-theory. Finally, an $Ł^{\leq}_{\infty}$-theory closed under the rule $p \vdash p * p$ is also closed under all rules $p \vdash p^{2^k}$ for all $k \geq 1$, by iteration. Since by $p^{2^k} \vdash p^n$ when $n \geq 2^k$ is also a rule of $Ł^{\leq}_{\infty}$, such theory is also closed under all rules $p \vdash p^n$ for all $n \geq 2$. That all these rules are proper follows from Theorem 11 and their mutual equivalence just established. ∎

Again, as in the finite case, I do not know of a Hilbert-style presentation of $Ł^{\leq}_{\infty}$, while a Gentzen-style one is proposed, without proof, in [22].

While the last results have been logical applications of the main results, I offer below an algebraic application of the logical results:

Proposition 26 *Let $\boldsymbol{A}$ be an MV-algebra, and let F be a lattice filter on $\boldsymbol{A}$. Then the largest implicative filter on $\boldsymbol{A}$ contained in F is the set $G = \{a \in A : a^k \in F$ for all $k \geq 1\}$.*

Proof: It is clear that $G \subseteq F$, because $a^1 = a$. In order to see that G is an $Ł^{\leq}_{\infty}$-filter, one assumes that $\varphi_0, \dots, \varphi_{n-1} \vdash^{\leq}_{\infty} \psi$ and that $h(\varphi_0), \dots, h(\varphi_{n-1}) \in G$ for some $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$; by definition of G this means that for all $k \geq 1$, $\big(h(\varphi_0)\big)^k, \dots, \big(h(\varphi_{n-1})\big)^k \in F$. Now by Proposition 10.1 also $\varphi_0^k, \dots, \varphi_{n-1}^k \vdash^{\leq}_{\infty} \psi^k$, and since F is an $Ł^{\leq}_{\infty}$-filter, one concludes that $h(\psi^k) \in F$ for all $k \geq 1$, that is, $h(\psi) \in G$. This shows that G is an $Ł^{\leq}_{\infty}$-filter. By its own definition G is trivially a model of the rule $p \vdash p * p$, hence by Theorem 25 it is an $Ł_{\infty}$-filter, that is, an implicative filter. Finally, let $H \subseteq F$ be another implicative filter, and $a \in H$. Again by Theorem 25 $a^k \in H$ for all $k \geq 1$, so $a^k \in F$ for all $k \geq 1$, which implies $a \in G$, that is, $H \subseteq G$. Thus G is the largest such implicative filter. ∎

The final observation on the logics examined in this paper is a relation between the finite and the infinite cases. At the end of Section 4 I promised to determine the lower bounds of the families of logics generated by the two schemes from finite subalgebras of $[\mathbf{0},\mathbf{1}]$, that is, of the families of all $Ł^{\leq}_{m}$ and of all the $Ł_m$. In doing this one finds an interesting fact: the desired infima are not the same if computed in the (natural) lattice of all logics over the same language, or if computed in the larger lattice of all consequence relations over the same language; such sets are complete lattices, see [44, Section 1.5].

Theorem 27 *The infimum of the family $\{Ł^{\leq}_{m} : m \geq 2\}$ in the lattice of all (finitary) logics is the logic $Ł^{\leq}_{\infty}$, while the infimum of the same family in the larger lattice of all consequence relations is a non-finitary logic different from $Ł^{\leq}_{\infty}$.*

Proof: This uses TARSKI's theorem [29, Theorem 20] that $\mathsf{V}\big(\{\boldsymbol{S}_m : m \geq 2\}\big) = \mathsf{V}\big([\mathbf{0},\mathbf{1}]\big) = \mathsf{MV}$; see also [8, Proposition 8.1.2]. Then the following chain of equivalences should be obvious:

$$
\begin{aligned}
&\varphi_0, \dots, \varphi_{n-1} \vdash_m^{\leq} \psi \quad \text{for all } m \geq 2 \\
&\iff \vdash_m \varphi_0 \wedge \dots \wedge \varphi_{n-1} \to \psi \quad \text{for all } m \geq 2 && \text{by (20)} \\
&\iff \varphi_0 \wedge \dots \wedge \varphi_{n-1} \to \psi \approx \top \ \text{ holds in } \boldsymbol{S}_m, \text{ for all } m \geq 2 \\
&\iff \varphi_0 \wedge \dots \wedge \varphi_{n-1} \to \psi \approx \top \ \text{ holds in } \mathsf{MV} && \text{by TARSKI's Theorem} \\
&\iff \vdash_\infty \varphi_0 \wedge \dots \wedge \varphi_{n-1} \to \psi && \text{by algebraizability} \\
&\iff \varphi_0, \dots, \varphi_{n-1} \vdash_\infty^{\leq} \psi && \text{by (24).}
\end{aligned}
$$

So $\text{Ł}_\infty^{\leq} = \inf\{\text{Ł}_m^{\leq} : m \geq 2\}$ in the lattice of (finitary) logics (over the same language, of course). Now put $\mathcal{S}_\infty = \inf_c\{\text{Ł}_m^{\leq} : m \geq 2\}$, the infimum of the same family in the bigger lattice of all consequence relations. That is, for all $\Gamma \subseteq Fm$ and all $\psi \in Fm$, $\Gamma \vdash_{\mathcal{S}_\infty} \psi$ if and only if for all $m \geq 2$, $\Gamma \vdash_m^{\leq} \psi$. Therefore, $\text{Ł}_\infty^{\leq}$ is the finitary part of $\mathcal{S}_\infty$, and so seeing that $\mathcal{S}_\infty$ is non-finitary and seeing that $\mathcal{S}_\infty \neq \text{Ł}_\infty^{\leq}$ amount to the same thing. Consider the set of formulas in two variables $\Delta(p,q) = \big\{(p \leftrightarrow q)^{m-1} : m \geq 2\big\}$. By Theorem 14, $p, (p \leftrightarrow q)^{m-1} \vdash_m^{\leq} q$ for each $m \geq 2$, hence a fortiori $p, \Delta(p,q) \vdash_m^{\leq} q$ for all $m \geq 2$, and thus $p, \Delta(p,q) \vdash_{\mathcal{S}_\infty} q$. However, $p, \Delta(p,q) \nvdash_\infty^{\leq} q$. In order to see this, since $\text{Ł}_\infty^{\leq}$ is finitary and $p^s \vdash p^t$ whenever $s \geq t \geq 1$, it is enough to see that $p, (p \leftrightarrow q)^{m-1} \nvdash_\infty^{\leq} q$ for each $m \geq 2$. For $m = 2$ it is enough to take $v(q) < v(p)$ while for $m > 2$ take $v(p) = 1/2$ and $\frac{m-3}{2m-4} < v(q) < 1/2$; a straightforward computation shows that in both cases $v(q) < v(p) \wedge v\big((p \leftrightarrow q)^{m-1}\big)$, which is against $p, (p \leftrightarrow q)^{m-1} \nvdash_\infty^{\leq} q$. ∎

Thus, this yields a beautiful, natural example that the lattice of all logics on a fixed language is not a *complete* sublattice of the lattice of all consequence relations over the same language; notice that by [44, Theorem 1.5.6] the former is indeed a sublattice of the latter. At the same time, since by Theorem 14 all $\text{Ł}_m^{\leq}$ are protoalgebraic while by Theorem 23 $\text{Ł}_\infty^{\leq}$ is not protoalgebraic, this also shows that the set of all protoalgebraic logics (over a fixed language) is not a complete sublattice of the lattice of all logics either. As the logics $\text{Ł}_m^{\leq}$ are all finitely equivalential, the same applies to the sets of all equivalential logics and of all finitely equivalential logics (always on the same language). However, the issue of the order structure of all these particular sets has not been touched in depth in the literature; it is however known that each of these sets is closed under strengthenings, see [11, pages 71,187,316].

Finally, almost the same situation is found for the strong versions:

Theorem 28 *The infimum of the family $\{\text{Ł}_m : m \geq 2\}$ in the lattice of all (finitary) logics is the logic Ł_∞, while the infimum of the same family in the larger lattice of all consequence relations is a non-finitary logic different from Ł_∞.*

Proof: The proof of the first part is similar to the first part of the proof of Theorem 27, but using a much more recent and sophisticated algebraic result. By definition of the logics Ł_m, the entailment $\varphi_0, \dots, \varphi_{n-1} \vdash_m \psi$ holds for all $m \geq 2$ if and only if the quasiequation

$$\varphi_0 \approx \top \ \& \ \dots \ \& \ \varphi_{n-1} \approx \top \ \Rightarrow \ \psi \approx \top \tag{26}$$

holds in $\boldsymbol{S}_m$ for all $m \geq 2$, that is, iff (26) holds in the quasivariety generated by the family of algebras $\{\boldsymbol{S}_m : m \geq 2\}$. Clearly, the union of all their universes contains all the rational points in $[0,1]$, hence as a consequence of Theorem 3.8 of [23] this quasivariety coincides with the variety generated by the same family, which by TARSKI's Theorem used in the preceding proof, is the whole class MV. Then, by the algebraizability of $Ł_\infty$ with MV as equivalent algebraic semantics, (26) holds in MV if and only if $\varphi_0, \ldots, \varphi_{n-1} \vdash_\infty \psi$. This proves that $Ł_\infty = \inf\{Ł_m : m \geq 2\}$. In order to prove the second part of the theorem one considers $\mathcal{S}_\infty = \inf_c\{Ł_m : m \geq 2\}$, the infimum of the same family in the bigger lattice of all consequence relations. To reach a contradiction let's assume that this consequence is finitary; then it belongs to the smaller lattice and hence $\mathcal{S}_\infty = Ł_\infty$. By definition $Ł_\infty$ is the finitary part of the consequence relation defined by the infinite matrix $\langle [\mathbf{0},\mathbf{1}], \{1\} \rangle$, which I denote by $\mathcal{S}^\infty$. As I recalled before and is well-known, $\mathcal{S}^\infty$ is not finitary, hence it is strictly stronger than its finitary part $Ł_\infty$. But for each $m \geq 2$, the matrix $\langle \boldsymbol{S}_m, \{1\} \rangle$ is a submatrix of $\langle [\mathbf{0},\mathbf{1}], \{1\} \rangle$, and by [11, Proposition 0.3.3] the consequence it defines is stronger than that defined by the larger matrix: Each $Ł_m$ is stronger than $\mathcal{S}^\infty$, hence so is their infimum in the bigger lattice, that is, $\mathcal{S}_\infty = Ł_\infty$ is stronger than $\mathcal{S}^\infty$, which contradicts the fact proved before that $\mathcal{S}^\infty$ is strictly stronger than $Ł_\infty$. This shows that $\mathcal{S}_\infty$ is non-finitary and hence different from $Ł_\infty$. ∎

The distinction between the greatest lower bounds of each family in the two different complete lattices does not carry over to their lowest upper bound: By Theorem 19 they coincide with $\mathcal{CPL} = Ł_2 = Ł_2^{\leq}$, which is a member of both families, hence it is actually their maximum inside the two lattices.

6 Conclusions

In this paper I have presented the theory of semilattice-based logics as a formalization, in the context of Abstract Algebraic Logic, of the (already known) idea of *preserving degrees of truth*, a scheme for semantic definition of logics particularly suited for multiple-valued logics opposed to the (more standard) scheme of *preservation of truth*. After summarizing the main notions and results in this theory, I have applied it, together with other more general parts of Abstract Algebraic Logic, to the study of the logics obtained in both ways from a single subalgebra of the real unit interval, endowed with Łukasiewicz's operations. The algebraic counterparts of these logics have been determined, and they have been classified according to the so-called protoalgebraic hierarchy. Some further properties have been obtained, either of an algebraic or of a logical character. The behaviour of the logics resulting from finite subalgebras has been seen as much more standard and regular than that of those arising from infinite subalgebras; in particular, the scheme of preservation of degrees of truth obtains only one logic in the latter case, while the scheme of preservation of truth obtains a different logic for each infinite subalgebra containing different sets of rationals from the real unit interval.

An interesting observation arises, that preservation of truth seems to depend on the rational points belonging to the defining truth-value algebra, while preservation of degrees of truth seems to do so only in case there are only a finite number of degrees of truth to be preserved. No interpretation of this fact has been attempted at.

The development of the last two sections of the paper shows many examples of how the interplay logic-algebra works, and in both directions. General theories of logic and of its algebraization have been used, as well as central notions of universal algebra. More importantly, several deep facts concerning the algebraic properties of the particular classes of algebras involved have been used in an essential way in order to obtain some of the main logical results.

I have given just a sample of how these methods work when dealing with logics defined from a subalgebra of the real unit interval. Clearly this does not exhaust the family of multiple-valued logics defined by preservation of degrees of truth in this context; actually, there is one such logic for each proper subvariety of the variety $\mathbf{MV}$ of all MV-algebras. By LINDENBAUM's and TARSKI's Theorems quoted before, such a subvariety can contain only a finite number of the subalgebras $\boldsymbol{S}_m$ and can contain no infinite subalgebra of $[\mathbf{0},\mathbf{1}]$; KOMORI [27] described all such subvarieties. The situation concerning subquasivarieties of $\mathbf{MV}$ (which are related with the logics obtained by preservation of truth) is far more complicated; those generated by families of subalgebras of $[\mathbf{0},\mathbf{1}]$ have been classified and described in [23].

The paper also shows the importance of having a general framework, as a way of placing the investigations on particular logics and/or particular classes of algebras in the context of a more far-reaching research program. I think that the contents of the paper shows that Abstract Algebraic Logic is a good candidate for playing such a role, and offers some powerful tools for these tasks.

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