CONVERGENCE OF TWO-STEP ITERATIVE SCHEME WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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A two-step iterative scheme with errors has been studied to approximate the common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence in Banach spaces.

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1. Introduction. In 1995, Liu \[4\] introduced iterative schemes with errors as follows.

(a) For a nonempty subset \(C\) of a normed space \(E\) and \(T : C \to C\), the sequence \(\{x_n\}\) in \(C\), iteratively defined by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nT y_n + u_n, \\
y_n &= (1 - b_n)x_n + b_nT x_n + v_n, \quad n \geq 1,
\end{align*}
\]

where \(\{a_n\}, \{b_n\}\) are sequences in \([0,1]\) and \(\{u_n\}, \{v_n\}\) are sequences in \(E\) satisfying \(\sum_{n=1}^{\infty} \|u_n\| < \infty\), \(\sum_{n=1}^{\infty} \|v_n\| < \infty\), is known as Ishikawa iterative scheme with errors.

(b) With \(E, C,\) and \(T\) as in (a), the sequence \(\{x_n\}\), iteratively defined by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nTx_n + u_n, \quad n \geq 1,
\end{align*}
\]

where \(\{a_n\}\) is a sequence in \([0,1]\) and \(\{u_n\}\) a sequence in \(E\) satisfying \(\sum_{n=1}^{\infty} \|u_n\| < \infty\), is known as Mann iterative scheme with errors.

In 1999, Huang \[2\] studied the above schemes for asymptotically nonexpansive mappings. Recall that a mapping \(T : C \to C\) is asymptotically nonexpansive if there is a sequence \(\{k_n\} \subset [1, \infty)\) with \(\lim_{n \to \infty} k_n = 1\) and \(\|T^n x - T^n y\| \leq k_n \|x - y\|\) for all \(x, y \in C\) and for all \(n \in \mathbb{N}\), where \(\mathbb{N}\) denotes the set of positive integers.

Moreover, in 2001, Khan and Takahashi \[3\] approximated the fixed points of two asymptotically nonexpansive mappings \(S, T : C \to C\) through the sequence \(\{x_n\}\) given by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= (1 - a_n)x_n + a_nS^n y_n, \\
y_n &= (1 - b_n)x_n + b_nT^n x_n,
\end{align*}
\]

where \(\{a_n\}, \{b_n\}\) are sequences in \([0,1]\) satisfying certain conditions.
Inspired and motivated by the study of the above schemes, we suggest a new iterative scheme \( \{x_n\} \) in \( C \) constructed through a pair of asymptotically nonexpansive mappings \( S, T : C \to C \) given by

\[
x_1 = x \in C, \\
x_{n+1} = (1 - a_n)x_n + a_nS^n y_n + u_n, \\
y_n = (1 - b_n)x_n + b_n T^n x_n + v_n, \quad n \geq 1,
\]

where \( \{a_n\} \), \( \{b_n\} \) are sequences in \([0, 1]\) with appropriate conditions and \( \{u_n\} \), \( \{v_n\} \) are sequences in \( E \) with \( \sum_{n=1}^{\infty} \|u_n\| < \infty \), \( \sum_{n=1}^{\infty} \|v_n\| < \infty \).

It is to be noted here that each of the above schemes follows as a special case of our scheme.

**2. Preliminaries.** Let \( E \) be a Banach space with \( C \) as its nonempty convex subset. Throughout this paper, \( \mathbb{N} \) denotes the set of positive integers and \( F(T) \) the set of fixed points of the mapping \( T \). Now we list the following definitions and results used to prove the results in the next section.

**Definition 2.1.** A mapping \( T : C \to C \) is uniformly \( k \)-Lipschitzian if for some \( k > 0 \),

\[
\|T^n x - T^n y\| \leq k\|x - y\| \quad \text{for all } x, y \in C \text{ and for all } n \in \mathbb{N}.
\]

**Definition 2.2.** A mapping \( T : C \to C \) is completely continuous if and only if \( \{Tx_n\} \) has a convergent subsequence for every bounded sequence \( \{x_n\} \) in \( C \).

**Definition 2.3.** \( E \) is said to satisfy Opial’s condition [5] if for any sequence \( \{x_n\} \) in \( E \), \( x_n \rightharpoonup x \) implies that

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \quad \text{for all } y \in E \text{ with } y \neq x.
\]

**Definition 2.4.** A mapping \( T : C \to E \) is called demiclosed with respect to \( y \in E \) if for each sequence \( \{x_n\} \) in \( C \) and each \( x \in E \), \( x_n \rightharpoonup x \) and \( Tx_n \to y \) imply that \( x \in C \) and \( Tx = y \).

**Lemma 2.5** [6]. Suppose that \( E \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \) for all \( n \in \mathbb{N} \). Also, suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that

\[
\limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r, \text{ and } \lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r \quad \text{hold for some } r \geq 0.
\]

Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 2.6** [7]. Let \( \{r_n\}, \{s_n\}, \{t_n\} \) be three nonnegative sequences satisfying

\[
r_{n+1} \leq (1 + s_n)r_n + t_n \quad \forall n \geq 1.
\]

If \( \sum_{n=1}^{\infty} s_n < \infty \) and \( \sum_{n=1}^{\infty} t_n < \infty \), then \( \lim_{n \to \infty} r_n \) exists.

**Lemma 2.7** [1]. Let \( E \) be a uniformly convex Banach space satisfying Opial’s condition and let \( C \) be a nonempty closed convex subset of \( E \). Let \( T \) be an asymptotically nonexpansive mapping of \( C \) into itself. Then \( I - T \) is demiclosed with respect to zero.
3. Approximating common fixed points. We start with the following lemma.

**Lemma 3.1.** Let $E$ be a normed space and $C$ a nonempty bounded closed convex subset of $E$. Let, for $k > 0$, $S$ and $T$ be uniformly $k$-Lipschitzian mappings of $C$ into itself. Let $\{x_n\}$ be a sequence as defined in (1.4), where $\{u_n\}$, $\{v_n\}$ are sequences in $E$ such that

$$\lim_{n \to \infty} \|u_n\| = 0 = \lim_{n \to \infty} \|v_n\|$$

and

$$\lim_{n \to \infty} \|x_n - S^n x_n\| = 0 = \lim_{n \to \infty} \|x_n - T^n x_n\|. \quad (3.1)$$

Then

$$\lim_{n \to \infty} \|x_n - S x_n\| = 0 = \lim_{n \to \infty} \|x_n - T x_n\|. \quad (3.2)$$

**Proof.** Take $c_n = \|x_n - T^n x_n\|$ and $d_n = \|x_n - S^n x_n\|$. Consider

$$\|x_{n+1} - x_n\| = \|a_n (S^{n+1} y_n - x_n) + u_n\|
\leq a_n \|S^n y_n - S^n x_n\| + \|S^n x_n - x_n\| + \|u_n\|
\leq a_n k \|(1 - b_n) x_n + b_n T^n x_n + v_n - x_n\| + a_n d_n + \|u_n\|
= a_n k \|b_n (T^n x_n - x_n) + v_n\| + a_n d_n + \|u_n\|
\leq a_n b_n c_n k + a_n k \|v_n\| + a_n d_n + \|u_n\|
\leq c_n k + d_n + k \|v_n\| + \|u_n\|.$$ \hspace*{1cm} (3.3)

That is,

$$\|x_{n+1} - x_n\| \leq c_n k + d_n + k \|v_n\| + \|u_n\|. \quad (3.4)$$

Next, consider

$$\|x_{n+1} - S x_{n+1}\| = \|(x_{n+1} - S^{n+1} x_{n+1}) + (S^{n+1} x_{n+1} - S x_{n+1})\|
\leq d_{n+1} + k \|(x_{n+1} - x_n) + (x_n - S^n x_n) + (S^n x_n - S^n x_{n+1})\|
\leq d_{n+1} + k d_n + k (k + 1) \|x_{n+1} - x_n\| \quad (3.5)
\leq d_{n+1} + k d_n + k (k + 1) [c_n k + d_n + k \|v_n\| + \|u_n\|]$$

by (3.4). Taking limsup on both sides in the above inequality, we obtain

$$\limsup_{n \to \infty} \|x_{n+1} - S x_{n+1}\| \leq 0. \quad (3.6)$$

That is,

$$\lim_{n \to \infty} \|x_n - S x_n\| = 0. \quad (3.7)$$

Similarly, we can prove that

$$\lim_{n \to \infty} \|x_n - T x_n\| = 0. \quad (3.8)$$

This completes the proof of the lemma. \hfill \Box
Lemma 3.2. Let $E$ be a uniformly convex Banach space and $C$ its nonempty bounded closed convex subset. Let $S$ and $T$ be self-mappings of $C$ satisfying

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad (3.9)$$

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $n \in \mathbb{N}$, where $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be as in (1.4) with $\{a_n\}, \{b_n\}$ in $[\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and $\{u_n\}, \{v_n\}$ in $E$ with $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n \to \infty} \|x_n - Sx_n\| = 0 = \lim_{n \to \infty} \|x_n - Tx_n\|$.

Proof. Let $p \in F(S) \cap F(T)$. Then

$$\|x_{n+1} - p\| = \|a_n (S^n y_n - p) + (1-a_n) (x_n - p) + u_n\|$$

$$\leq a_n k_n \|y_n - p\| + (1-a_n) \|x_n - p\| + \|u_n\|$$

$$= a_n k_n \|(1-b_n)x_n + b_n T^n x_n + v_n - p\| + (1-a_n) \|x_n - p\| + \|u_n\|$$

$$= a_n k_n \|b_n (T^n x_n - p) + (1-b_n) (x_n - p) + v_n\| + (1-a_n) \|x_n - p\| + \|u_n\|$$

$$\leq a_n b_n k_n \|x_n - p\| + a_n k_n \|v_n\| + a_n (1-b_n)k_n \|x_n - p\| + (1-a_n) \|x_n - p\| + \|u_n\|$$

$$= (1 + a_n b_n k_n^2 + a_n (1-b_n)k_n - a_n) \|x_n - p\| + a_n k_n \|v_n\| + \|u_n\|.$$ 

(3.10)

Since $\{k_n\}$ is a bounded sequence, therefore there exists $h > 0$ such that $k_n \leq h$ for all $n \geq 1$ so that

$$\|x_{n+1} - p\| \leq \left[ 1 + a_n b_n h (k_n - 1) + a_n (k_n - 1) \right] \|x_n - p\| + a_n h \|v_n\| + \|u_n\|.$$ 

(3.11)

Take $s_n = a_n b_n h (k_n - 1) + a_n (k_n - 1)$, $t_n = a_n h \|v_n\| + \|u_n\|$, and $r_n = \|x_n - p\|$. As $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, so $\lim_{n \to \infty} \|x_n - p\|$ exists by Lemma 2.6. Let $\lim_{n \to \infty} \|x_n - p\| = c$, where $c \geq 0$ is a real number. Assume that $c > 0$, as the result for the case $c = 0$ is obviously true. Now $\|T^n x_n - p\| \leq k_n \|x_n - p\|$ for all $n \in \mathbb{N}$ gives $\limsup_{n \to \infty} \|T^n x_n - p\| \leq c$. Also,

$$\|y_n - p\| = \|b_n (T^n x_n - p) + (1-b_n) (x_n - p) + v_n\|$$

$$\leq \|x_n - p\| + (k_n - 1)b_n \|x_n - p\| + \|v_n\|$$ 

(3.12)

gives

$$\limsup_{n \to \infty} \|y_n - p\| \leq c.$$ 

(3.13)

Next, consider

$$\|S^n y_n - p + a_n^{-1} u_n\| \leq k_n \|y_n - p\| + a_n^{-1} \|u_n\| \leq k_n \|y_n - p\| + \frac{1}{\delta} \|u_n\|.$$ 

(3.14)
By the above inequality and by virtue of \( \|u_n\| \to 0 \) and \( k_n \to 1 \) as \( n \to \infty \), we get
\[
\limsup_{n \to \infty} \|S^n y_n - p + a_n^{-1} u_n\| \leq c. \tag{3.15}
\]
Moreover, \( c = \lim_{n \to \infty} \|x_{n+1} - p\| \) means that
\[
\lim_{n \to \infty} \|a_n (S^n y_n - p + a_n^{-1} u_n) + (1 - a_n) (x_n - p)\| = c. \tag{3.16}
\]
Applying Lemma 2.5,
\[
\lim_{n \to \infty} \|S^n y_n - x_n + a_n^{-1} u_n\| = 0. \tag{3.17}
\]
Thus
\[
\|S^n y_n - x_n\| \leq \|S^n y_n - x_n + a_n^{-1} u_n\| + \frac{1}{\delta} \|u_n\| \tag{3.18}
\]
yields that
\[
\lim_{n \to \infty} \|S^n y_n - x_n\| = 0. \tag{3.19}
\]
Also, then
\[
\|x_n - p\| \leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \leq \|x_n - S^n y_n\| + k_n \|y_n - p\| \tag{3.20}
\]
implies that
\[
c \leq \liminf_{n \to \infty} \|y_n - p\|. \tag{3.21}
\]
By (3.13) and (3.21), we obtain
\[
\lim_{n \to \infty} \|y_n - p\| = c. \tag{3.22}
\]
That is,
\[
\lim_{n \to \infty} \|b_n (T^n x_n - p + b_n^{-1} v_n) + (1 - b_n) (x_n - p)\| = c. \tag{3.23}
\]
Again by Lemma 2.5, we get
\[
\lim_{n \to \infty} \|T^n x_n - x_n + b_n^{-1} v_n\| = 0, \tag{3.24}
\]
which finally gives that
\[
\lim_{n \to \infty} \|T^n x_n - x_n\| = 0. \tag{3.25}
\]
Now
\[
\|S^n x_n - x_n\| \leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\|
\leq k_n b_n \|T^n x_n - x_n\| + \|v_n\| + \|S^n y_n - x_n\| \tag{3.26}
\]
implies, together with (3.19) and (3.25), that

$$\lim_{n \to \infty} \| S^n x_n - x_n \| = 0 = \lim_{n \to \infty} \| T^n x_n - x_n \|. \quad (3.27)$$

Lemma 3.1 now reveals that

$$\lim_{n \to \infty} \| S x_n - x_n \| = 0 = \lim_{n \to \infty} \| T x_n - x_n \|, \quad (3.28)$$

which is as desired.

**Theorem 3.3.** Let $E$ be a uniformly convex Banach space satisfying Opial’s condition and let $S$, $T$, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of $S$ and $T$.

**Proof.** Let $p \in F(S) \cap F(T)$. Then, as proved in Lemma 3.2, $\lim_{n \to \infty} \| x_n - p \|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. To prove this, let $w_1$ and $w_2$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.2, $\lim_{n \to \infty} \| x_n - S x_n \| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 2.7; therefore, we obtain $S w_1 = w_1$. Similarly, $T w_1 = w_1$. Again, in the same way, we can prove that $w_2 \in F(S) \cap F(T)$. Next, we prove the uniqueness. For this, suppose that $w_1 \neq w_2$; then by Opial’s condition,

$$\lim_{n \to \infty} \| x_n - w_1 \| = \lim_{n_i \to \infty} \| x_{n_i} - w_1 \| < \lim_{n_i \to \infty} \| x_{n_i} - w_2 \| = \lim_{n \to \infty} \| x_n - w_2 \| = \lim_{n_j \to \infty} \| x_{n_j} - w_2 \|$$

$$< \lim_{n_j \to \infty} \| x_{n_j} - w_1 \| = \lim_{n \to \infty} \| x_n - w_1 \|, \quad (3.29)$$

a contradiction. Hence the proof is over.

**Remark 3.4.** If we take $u_n = v_n = 0$ for all $n \in \mathbb{N}$, the above theorem reduces to [3, Theorem 1] of Khan and Takahashi. Moreover, [6, Theorem 2.1] of Schu becomes a special case of the above theorem when $u_n = v_n = 0$ as well as $T = I$, the identity mapping.

Finally, we approximate common fixed points by the following strong convergence theorem.

**Theorem 3.5.** Let $E$ be a uniformly convex Banach space and $C$ its bounded closed convex subset. Let $S$, $T$, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S) \cap F(T) \neq \emptyset$ and either $S$ or $T$ is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.

**Proof.** Assume that $T : C \to C$ is completely continuous. Since $\{x_n\}$ is a bounded sequence and $T$ is completely continuous, therefore $\{T x_n\}$ must have a convergent subsequence $\{T x_{n_i}\}$. Hence by (3.28), $\{x_n\}$ must have a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q$ (say) in $C$ as $n_i \to \infty$. Now continuity of $S$ and $T$ gives that $S x_{n_i} \to S q$ and $T x_{n_i} \to T q$ as $n_i \to \infty$. Then, again by (3.28), $\| S q - q \| = 0 = \| T q - q \|$. This yields that $q \in F(S) \cap F(T)$ so that $\{x_{n_i}\}$ converges strongly to $q$ in $F(S) \cap F(T)$. As proved in
Lemma 3.2, \( \lim_{n \to \infty} \| x_n - p \| \) exists for all \( p \in F(S) \cap F(T) \); therefore, \( \{ x_n \} \) must itself converge to \( q \in F(S) \cap F(T) \). Hence the proof. \( \square \)

**Remark 3.6.** If we put \( T = I, v_n = 0 \) in the above theorem, then [2, Theorem 1] of Huang is obtained. When we take \( S = T \) in the above theorem, then [2, Theorem 2] of Huang follows except when \( b_n = 0 \). Since a self-mapping with compact domain is completely continuous, therefore [3, Theorem 2] of Khan and Takahashi can also be obtained by putting \( u_n = v_n = 0 \). It is also worth mentioning that the results presented in this paper are for two mappings while the results in Huang [2] are for one mapping only. Meanwhile, calculations in this paper are made much simpler as compared to Huang [2].

**References**


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