COEFFICIENT ESTIMATES FOR RUSCHEWEYH DERIVATIVES

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We consider functions $f$, analytic in the unit disc and of the normalized form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For functions $f \in \bar{R}_\delta(\beta)$, the class of functions involving the Ruscheweyh derivatives operator, we give sharp upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$.

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1. Introduction. Let $S$ denote the class of normalized analytic univalent functions $f$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the unit disc $D = \{ z : |z| < 1 \}$. Suppose that

$$S^* = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \, z \in D \right\},$$

$$S^*(\beta) = \left\{ f \in S : \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta \pi}{2}, \, z \in D \right\} \quad (1.2)$$

are classes of starlike and strongly starlike functions of order $\beta \ (0 < \beta \leq 1)$, respectively. Note that $S^*(\beta) \subset S^*$ for $0 < \beta < 1$ and $S^*(1) = S^*$ [5]. Kanas [2] introduced the subclass $\bar{R}_\delta(\beta)$ of function $f \in S$ as the following.

**Definition 1.1.** For $\delta \geq 0$, $\beta \in (0,1]$, a function $f$ normalized by (1.1) belongs to $\bar{R}_\delta(\beta)$ if, for $z \in D \setminus \{0\}$ and $D^\delta f(z) \neq 0$, the following holds:

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| \leq \frac{\beta \pi}{2}, \quad (1.3)$$

where $D^\delta f$ denotes the generalized Ruscheweyh derivative which was originally defined as the following.

**Definition 1.2** [6]. Let $D^n f$ and $f$ be defined by (1.1). Then for $n \in \mathbb{N} \cup \{0\}$,

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} \ast f(z), \quad (1.4)$$

where $\ast$ denotes the Hadamard product of two analytic functions and $\mathbb{N}$ is a set of natural numbers.
Later in [1], Al-Amiri generalized the Ruscheweyh derivative \( D^\delta \) for real numbers \( \delta \geq -1 \) as a Hadamard product of the functions \( f \) and \( z/(1-z)^{\delta+1} \).

Note that \( \bar{R}_0(\beta) = S^*(\beta) \) for each \( \beta \in (0,1] \) and \( \bar{R}_0(1) = S^* \). In this note, we obtain sharp estimates for \( |a_2|, |a_3| \) and the Fekete-Szegö functional for the class \( \bar{R}_\delta(\beta) \). For the subclass \( S^* \), sharp upper bounds for the functional \( |a_3 - \mu a_2^2| \) have been obtained for all real \( \mu \) [3, 4].

2. Preliminary results. In proving our results, we will need the following lemmas. However, we first denote \( P \) to be the class of analytic functions with positive real part in \( D \).

**Lemma 2.1.** Let \( p \in P \) and let it be of the form \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) with \( \text{Rep}(z) > 0 \). Then

(i) \( |c_n| \leq 2 \) for \( n \geq 1 \),
(ii) \( |c_2 - c_1^2/2| \leq 2 - |c_1|^2/2 \).

**Lemma 2.2.** Let \( \delta \geq 0 \) and \( \beta \in (0,1] \). If \( f \in \bar{R}_\delta(\beta) \) and is given by (1.1), then

\[
|a_2| \leq \frac{2\beta}{\delta+1},
|a_3| \leq \begin{cases} 
\frac{2\beta}{(\delta+2)(\delta+1)} & \text{if } \beta \leq \frac{1}{3}, \\
\frac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \frac{1}{3}.
\end{cases}
\]

**Proof.** Let \( F(z) = D^\delta f(z) = z + A_2 z^2 + A_3 z^3 + \cdots \). Since \( f \in \bar{R}_\delta(\beta) \) and \( D^\delta f(z) \in S^*(\beta) \), then

\[
\frac{z F'(z)}{F(z)} = p^\beta(z)
\]

and so

\[
\frac{z(1 + 2A_2 z + 3A_3 z^2 + \cdots)}{z + A_2 z^2 + A_3 z^3 + \cdots} = (1 + c_1 z + c_2 z^2 + \cdots)^\beta,
\]

which implies that

\[
z + 2A_2 z^2 + 3A_3 z^3 + \cdots = z + (\beta c_1 + A_2) z^2 + \left( \beta c_2 + \frac{\beta(\beta-1)}{2} c_1^2 + \beta A_2 c_1 + A_3 \right) z^3 + \cdots.
\]

Equating the coefficients, we have

\[
A_2 = \beta c_1, \quad (2.5)
\]
\[
A_3 = \frac{\beta}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2.
\]

(2.6)
Now, for $\delta \geq -1$, $D^\delta f$ has the Taylor expansion

$$D^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(1+\delta)} a_n z^n, \quad z \in D,$$

(2.7)

where $\Gamma(n+\delta)$ denotes Euler’s functions with

$$\Gamma(n+\delta) = \delta(\delta+1) \cdots (\delta+n-1) \Gamma(\delta).$$

(2.8)

Then

$$z + A_2 z^2 + A_3 z^3 + \cdots = z + \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} a_2 z^2 + \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} a_3 z^3 + \cdots.$$

(2.9)

Equating the coefficients in (2.9), we have

$$a_2 \frac{\Gamma(2+\delta)}{\Gamma(1+\delta)} = a_2 (\delta+1) = A_2.$$

(2.10)

Then, from (2.5), we obtain

$$a_2 = \frac{\beta c_1}{\delta+1}.$$

(2.11)

It follows that from Lemma 2.1(i)

$$|a_2| \leq \frac{2\beta}{\delta+1},$$

(2.12)

whereas the coefficient of $z^3$ in (2.9) is

$$a_3 \frac{\Gamma(3+\delta)}{2\Gamma(1+\delta)} = a_3 \frac{(\delta+1)(\delta+2)}{2} = A_3.$$

(2.13)

From (2.6), we obtain

$$a_3 = \frac{2}{(\delta+1)(\delta+2)} \left[ \frac{\beta}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{3}{4} \beta^2 c_1^2 \right].$$

(2.14)

It follows from Lemma 2.1(ii) that

$$|a_3| \leq \frac{2}{(\delta+1)(\delta+2)} \left[ \frac{\beta}{2} \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{3}{4} \beta^2 |c_1|^2 \right],$$

(2.15)

that is,

$$|a_3| \leq \begin{cases} \frac{2\beta}{(\delta+1)(\delta+2)} & \text{if } \beta \leq \frac{1}{3}, \\ \frac{6\beta^2}{(\delta+2)(\delta+1)} & \text{if } \beta \geq \frac{1}{3}. \end{cases}$$

(2.16)
3. Results. We first consider the functional $|a_3 - \mu a_2^2|$ for complex $\mu$.

**Theorem 3.1.** Let $f \in \bar{R}_\delta(\beta)$ and $\beta \in (0, 1]$. Then for $\mu$ complex,

$$|a_3 - \mu a_2^2| \leq \frac{2\beta}{(\delta + 1)(\delta + 2)} \max \left[1, \frac{\beta(3(\delta + 1) - 2\mu(\delta + 2))}{(\delta + 1)} \right].$$  \hspace{1cm} (3.1)

For each $\mu$ there is a function in $\bar{R}_\delta(\beta)$ such that equality holds.

**Proof.** From (2.11) and (2.14), we write

$$a_3 - \mu a_2^2 = \frac{2}{(\delta + 1)(\delta + 2)} \left[ \frac{\beta}{2} (c_2 - \frac{c_1^2}{2}) + \frac{3}{4} \beta^2 c_1^2 \right] - \mu \left( \frac{\beta c_1}{\delta + 1} \right)^2,$$

$$= \frac{1}{(\delta + 1)(\delta + 2)} \left[ \beta \left( c_2 - \frac{c_1^2}{2} \right) + \beta^2 \left( \frac{3(\delta + 1) - 2\mu(\delta + 2)}{2(\delta + 1)^2(\delta + 2)} c_1^2 \right) \right].$$  \hspace{1cm} (3.2)

It follows from (3.2) and Lemma 2.1(ii) that

$$|a_3 - \mu a_2^2| \leq \frac{\beta}{(\delta + 1)(\delta + 2)} \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{\beta^2 (3(\delta + 1) - 2\mu(\delta + 2))}{2(\delta + 1)^2(\delta + 2)} |c_1|^2,$$

$$= \frac{2\beta}{(\delta + 1)(\delta + 2)} + \frac{\beta^2 (3(\delta + 1) - 2\mu(\delta + 2)) - \beta(\delta + 1)}{2(\delta + 1)^2(\delta + 2)} |c_1|^2,$$  \hspace{1cm} (3.3)

which on using Lemma 2.1(i), that is, $|c_1| \leq 2$, gives

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{2\beta}{(\delta + 1)(\delta + 2)} & \text{if } \kappa(\delta) \leq \beta(\delta + 1), \\
\frac{\beta^2 (6(\delta + 1) - 4\mu(\delta + 2))}{(\delta + 1)^2(\delta + 2)} & \text{if } \kappa(\delta) \geq \beta(\delta + 1),
\end{cases}$$  \hspace{1cm} (3.4)

where $\kappa(\delta) = |\beta^2 (3(\delta + 1) - 2\mu(\delta + 2))|$. Equality is attained for functions in $\bar{R}_\delta(\beta)$ given by

$$\frac{z(D^\delta f(z))'}{D^\delta f(z)} = \left( \frac{1 + z^2}{1 - z^2} \right)^\beta, \quad \frac{z(D^\delta f(z))'}{D^\delta f(z)} = \left( \frac{1 + z}{1 - z} \right)^\beta,$$  \hspace{1cm} (3.5)

respectively.

We next consider the cases where $\mu$ is real and prove the following.
**Theorem 3.2.** Let \( f \in R_\delta(\beta) \) and \( \beta \in (0, 1] \). Then for \( \mu \) real,

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\beta^2(6(\delta + 1) - 4\mu(\delta + 2))}{(\delta + 1)^2(\delta + 2)} & \text{if } \mu \leq \frac{(6\beta - 2)(\delta + 1)}{4\beta(\delta + 2)}, \\
\frac{2\beta}{(\delta + 1)(\delta + 2)} & \text{if } \frac{(6\beta - 2)(\delta + 1)}{4\beta(\delta + 2)} \leq \mu \leq \frac{2 + 6\beta)(\delta + 1)}{4\beta(\delta + 2)}, \\
\frac{\beta^2(4\mu(\delta + 2) - 6(\delta + 1))}{(\delta + 1)^2(\delta + 2)} & \text{if } \mu \geq \frac{(2 + 6\beta)(\delta + 1)}{4\beta(\delta + 2)}.
\end{cases}
\] (3.6)

For each \( \mu \), there is a function in \( R_\delta(\beta) \) such that equality holds.

**Proof.** Here we consider two cases.

Case (i): \( \mu \leq 3(\delta + 1)/2(\delta + 2) \).

In this case, (3.2) and Lemma 2.1(ii) give

\[
|a_3 - \mu a_2^2| \leq \frac{\beta}{(\delta + 1)(\delta + 2)} \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{\beta^2(6(\delta + 1) - 4\mu(\delta + 2))}{4(\delta + 1)^2(\delta + 2)} |c_1|^2,
\]

and so, using the fact that \( |c_1| \leq 2 \), we obtain

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\beta^2(6(\delta + 1) - 4\mu(\delta + 2))}{(\delta + 1)^2(\delta + 2)} & \text{if } \mu \leq \frac{(6\beta - 2)(\delta + 1)}{4\beta(\delta + 2)}, \\
\frac{2\beta}{(\delta + 1)(\delta + 2)} & \text{if } \frac{(6\beta - 2)(\delta + 1)}{4\beta(\delta + 2)} \leq \mu \leq \frac{3(\delta + 1)}{2(\delta + 2)}.
\end{cases}
\] (3.8)

Equality is attained on choosing \( c_1 = c_2 = 2 \) and \( c_1 = 0, c_2 = 2 \), respectively, in (3.2).

Case (ii): \( \mu \geq 3(\delta + 1)/2(\delta + 2) \).

It follows from (3.2) and Lemma 2.1(ii) that

\[
|a_3 - \mu a_2^2| \leq \frac{\beta}{(\delta + 1)(\delta + 2)} \left( 2 - \frac{|c_1|^2}{2} \right) + \frac{\beta^2(4\mu(\delta + 2) - 6(\delta + 1))}{4(\delta + 1)^2(\delta + 2)} |c_1|^2,
\]

and so, using the fact that \( |c_1| \leq 2 \), we obtain

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{2\beta}{(\delta + 1)(\delta + 2)} & \text{if } \frac{3(\delta + 1)}{2(\delta + 2)} \leq \mu \leq \frac{(6\beta + 2)(\delta + 1)}{4\beta(\delta + 2)}, \\
\frac{\beta^2(4\mu(\delta + 2) - 6(\delta + 1))}{(\delta + 1)^2(\delta + 2)} & \text{if } \mu \leq \frac{(6\beta + 2)(\delta + 1)}{4\beta(\delta + 2)}.
\end{cases}
\] (3.10)

Equality is attained on choosing \( c_1 = 0, c_2 = 2 \) and \( c_1 = 2i, c_2 = -2 \), respectively, in (3.2). Thus the proof is complete. □
Theorem 3.3. Let \( f \in \mathcal{R}_\delta(\beta) \) and let it be given by (1.1). Then

\[
|a_3| - |a_2| \leq \frac{2\beta}{(\delta + 1)(\delta + 2)} \quad \text{if } \beta \leq \frac{3(\delta + 1)}{5\delta + 1}.
\]

Proof. Write

\[
|a_3| - |a_2| \leq |a_3 - \frac{2}{3} a_2^2| + \frac{2}{3} |a_2|^2 - |a_2|.
\]

Then since \((6\beta - 2)(\delta + 1)/4\beta(\delta + 2) \leq 2/3 \) for \( \beta \leq 3(\delta + 1)/(5\delta + 1) \), it follows from Theorem 3.2 that

\[
|a_3| - |a_2| \leq \frac{2\beta}{(\delta + 1)(\delta + 2)} + \frac{2}{3} |a_2|^2 - |a_2| = \lambda(x),
\]

where \( x = |a_2| \in [0, 2\beta/(\delta + 1)] \). Since \( \lambda(x) \) attains its maximum value at \( x = 0 \), the theorem follows. This is sharp.

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References


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